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DISTORTIONS AND STRESSES OF PARABOLOIDAL SURFACE STRUCTURES

Part II -- The Membrane Behavior

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Frederic Y. M. Wan

4 January 1963

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The paraboloidal shell will, under the proper circumstances, prefer to carry the applied loads by the development of the force resultants $N_{\alpha\beta}$ rather than by the development of the transverse shear resultants Q_{α} and the bending moment resultants, $M_{\alpha\beta}$. Such behavior is commonly called membrane behavior although the term "Momentless" behavior is the more connotative. The membrane or momentless behavior will be the dominant mode of behavior when either the bending moments induced by the loads are negligibly small or whenever the flexural (bending) rigidity of the shell is sufficiently small so that the shell will deform without inducing appreciable bending moments. In this chapter, various solutions of the membrane equations as written in terms of the polar parameters will be presented. The so-called membrane equations are easily obtained by omitting the $M_{\alpha\beta}$, Q_{α} ,and $R_{\alpha\beta}$ terms from the equations which have been derived in the previous sections.

6.1 THE GOVERNING EQUATIONS

The equations which govern the membrane behavior of a paraboloidal shell will be summarized in this section.

The strain-displacement relations (see equations 3.4.19, 3.4.20, 3.4.21):

$$\epsilon_r^o = \frac{1}{2f\sqrt{1+(\gamma)^2}} \frac{\partial u_r^o}{\partial \gamma} - \frac{w}{2f\left[1+(\gamma)^2\right]^{3/2}} , \qquad 6.1.1$$

$$\epsilon_\theta^o = \frac{1}{2f\gamma} \frac{\partial u_\theta^o}{\partial \theta} + \frac{u_r^o}{2f\gamma\sqrt{1+(\gamma)^2}} - \frac{w}{2f\sqrt{1+(\gamma)^2}} , \qquad 6.1.2$$

$$2\epsilon_{r\theta}^o = \frac{1}{2f\sqrt{1+(\gamma)^2}} \frac{\partial u_\theta^o}{\partial \gamma} - \frac{u_\theta^o}{2f\gamma\sqrt{1+(\gamma)^2}} + \frac{1}{2f\gamma} \frac{\partial u_r^o}{\partial \theta} . \qquad 6.1.3$$

The equations of equilibrium (see equations 4.4.1.14, 4.4.1.15, 4.4.1.16):

$$\gamma \frac{\partial N_r}{\partial \gamma} + \sqrt{1+(\gamma)^2} \frac{\partial N_{r\theta}}{\partial \theta} + N_r - N_\theta + 2f\gamma\sqrt{1+(\gamma)^2}\, p_r = 0, \qquad 6.1.4$$

$$\gamma \frac{\partial N_{r\theta}}{\partial \gamma} + \sqrt{1+(\gamma)^2} \frac{\partial N_\theta}{\partial \theta} + 2N_{r\theta} + 2f\gamma\sqrt{1+(\gamma)^2}\, p_\theta = 0, \qquad 6.1.5$$

$$\frac{1}{1+(\gamma)^2} N_r + N_\theta + 2f\sqrt{1+(\gamma)^2} \quad P_n = 0 . \qquad 6.1.6$$

The stress-strain relations (see equations 5.2.7, 5.2.8, 5.2.9)

$$N_r = \frac{Eh}{1-\nu^2}\left(\epsilon_r^o + \nu\,\epsilon_\theta^o\right), \qquad 6.1.7$$

$$N_{r\theta} = \frac{Eh}{1-\nu^2}\left(1-\nu\right)\epsilon_{r\theta}^o , \qquad 6.1.8$$

$$N_\theta = \frac{Eh}{1-\nu^2}\left(\epsilon_\theta^o + \nu\,\epsilon_r^o\right). \qquad 6.1.9$$

There are nine equations for the nine unknown quantities:

$$\epsilon_r^o,\ \epsilon_{r\theta}^o,\ \epsilon_\theta^o,\ u_r^o,\ u_\theta^o,\ w,\ N_r,\ N_{r\theta},\ N_\theta .$$

6.2 THE BOUNDARY CONDITIONS

We will be concerned in the main with a paraboloidal shell which is bounded by one or two boundaries located at $\gamma = \gamma_1$ and $\gamma = \gamma_2$, i.e., the boundaries are circles of latitutde. These possibilities are shown in figure 6.2.1.

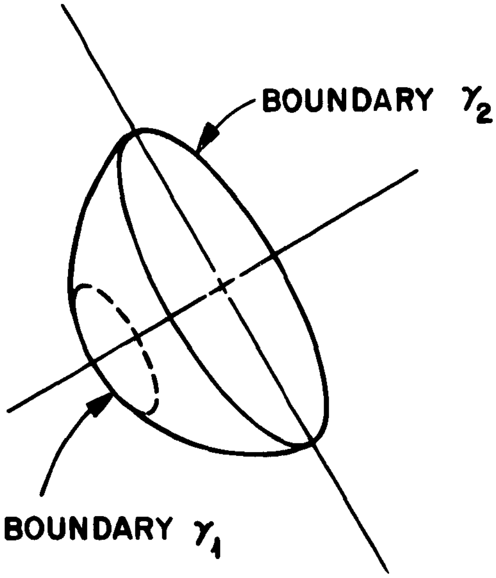


Figure 6.2.1

Note that the case where the shell is closed at the apex is not excluded here as γ_1 can well be zero.

The system of differential equations for determining the displacements is a second order system (cf. 6.1.1, 6.1.2 and 6.1.3). However, the stress resultants entering into the right hand sides of these differential equations are themselves solutions of a second order system (cf. 6.1.4, 6.1.5 and 6.1.6). Hence the displacements in the membrane theory satisfy a fourth order system of equations. Correspondingly, two independent boundary conditions must be given at each of the edges of the shell. It is clear from the equations themselves that half of these conditions must be given in terms of the displacements while the remaining conditions can be given either in terms of the displacements or the stress resultants.

It should be observed that the number of boundary conditions which can be

satisfied in the membrane theory is only half of that satisfied in the general theory. This phenomenon stems from the basic assumption in membrane theory that the shell has no bending stiffness in which case the unknowns $M_{\alpha\beta}$, Q_α, and $k_{\alpha\beta}$ need not be considered.

In the case of a shell with both edges restrained against tangential motion, the boundary conditions are all given in terms of the displacements. They are

$$u_r^\circ(\gamma_1, \theta) = 0, \tag{6.2.1}$$

$$u_r^\circ(\gamma_2, \theta) = 0, \tag{6.2.2}$$

$$u_\theta^\circ(\gamma_1, \theta) = 0, \tag{6.2.3}$$

$$u_\theta^\circ(\gamma_2, \theta) = 0. \tag{6.2.4}$$

On the other hand, the boundary conditions for a shell with one edge free, γ_1, and the other edge, γ_2, restrained against tangential motion are

$$N_r(\gamma_1, \theta) = 0, \tag{6.2.5}$$

$$N_{r\theta}(\gamma_1, \theta) = 0, \tag{6.2.6}$$

$$u_r^\circ(\gamma_2, \theta) = 0, \tag{6.2.7}$$

$$u_\theta^\circ(\gamma_2, \theta) = 0. \tag{6.2.8}$$

If the shell is closed at the apex, the usual procedure is to require that the stress resultants be finite there.

It will be shown in a subsequent section that, under the conditions enumerated in equations 6.2.1 to 6.2.8, the shell exhibits no in-extensional deformations (i.e. deformations without straining the middle surface of the shell). The corresponding membrane analysis will be simplified considerably.

6.3 THE LOADS DUE TO GRAVITY

A type of loading which is of great interest is that which is caused by the dead weight of the shell, i.e., the so-called gravity loading. In this section we will express the loading intensities p_r, p_θ, p_n in terms of the gravity loading. Consider the case where the shell has rotated from its face-up position (see figure 6.3.1) though an angle ψ.

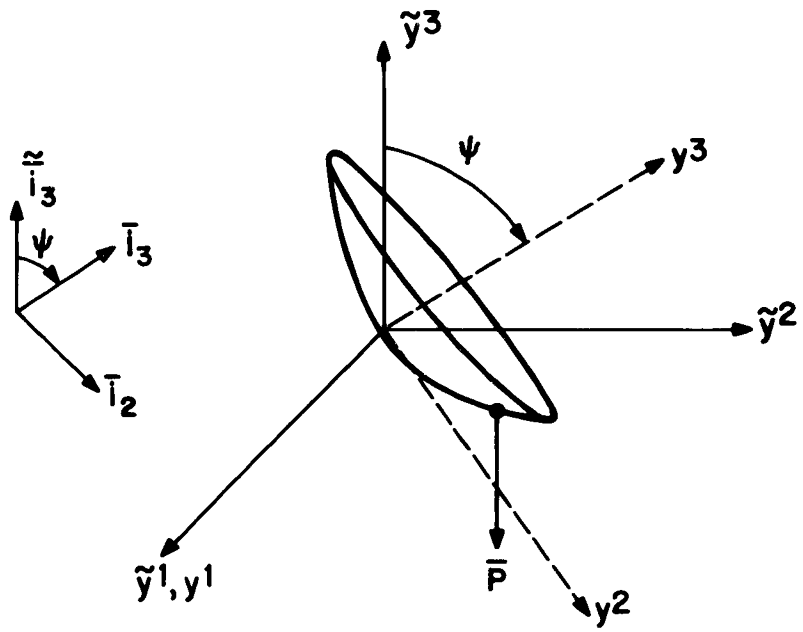


Figure 6.3.1

The $\tilde{y}^n$ axes are fixed in space such that the gravity axis is in the negative $\tilde{y}^3$ direction. Without loss of generality, we assume the shell to have rotated about its own y^1 axis which coincides with the $\tilde{y}^1$ axis. Let the load intensity vector due to the force of gravity be denoted by $\bar{\rho}$. Then

$$\bar{\rho} = -\rho_0 h \tilde{\tilde{i}}_3 \tag{6.3.1}$$

where ρ_0 is the weight density of the material in the shell (units of pounds per unit volume), h is the thickness, and $\tilde{\bar{i}}_3$ is the unit vector in the $\tilde{y}^3$ direction. With respect to the coordinates y^n which are fixed in the shell, we have the simple relationship

$$\tilde{\bar{i}}_3 = -\bar{i}_2 \sin\psi + \bar{i}_3 \cos\psi \qquad 6.3.2$$

where $\bar{i}_2$ and $\bar{i}_3$ are the unit vectors associated with y^2 and y^3. Thus

$$\bar{p} = \rho_0 h \sin\psi\, \bar{i}_2 - \rho_0 h \cos\psi\, \bar{i}_3 . \qquad 6.3.3$$

In the shell coordinates (see equation 4.4.1.19) the load intensity vector is expressed in terms of the coordinates on the middle surface of the shell.

$$\bar{p} = p_r \frac{\bar{a}_1}{\sqrt{a_{11}}} + p_\theta \frac{\bar{a}_2}{\sqrt{a_{22}}} + p_n \bar{n} \qquad 6.3.4$$

where

$$p_r = \bar{p} \cdot \frac{\bar{a}_1}{\sqrt{a_{11}}} = \frac{\rho_0 h}{\sqrt{1+(\gamma)^2}} \left[\sin\theta \sin\psi - \gamma\cos\psi\right], \qquad 6.3.5$$

$$p_\theta = \bar{p} \cdot \frac{\bar{a}_2}{\sqrt{a_{22}}} = \rho_0 h \cos\theta \sin\psi , \qquad 6.3.6$$

$$p_n = \bar{p} \cdot \bar{n} = \left[-\gamma \sin\theta \sin\psi - \cos\psi \right] \frac{\rho_0 h}{\sqrt{1+(\gamma)^2}} \qquad 6.3.7$$

In obtaining equations 6.3.5, 6.3.6, and 6.3.7, we have made use of the relations connecting the base vectors $\bar{a}_1$, $\bar{a}_2$, $\bar{n}$, and $\bar{i}_2$, $\bar{i}_3$, (see equations 2.1.8, 2.1.9, 2.1.14a).

6.4 THE AXI-SYMMETRIC MEMBRANE BEHAVIOR: GRAVITY LOADS

If the structural configuration of the shell is axi-symmetric and if the loading is also axi-symmetric as in the case of the loading due to gravity, then the membrane equations become further simplified. The axi-symmetric problem is characterized by

$$\frac{\partial}{\partial\theta} = 0 , \qquad 6.4.1$$

$$u_\theta^0 = 0, \qquad 6.4.2$$

$$p_\theta = 0 \qquad 6.4.3$$

and the remaining partial derivative becomes an ordinary derivative, $\frac{d}{d\gamma}$.

For the validity of this statement and the uniqueness of the solution

sought, readers are referred to section 6.6.

Under these restrictions the strain-displacement relations become (see equations 6.1.1, 6.1.2, 6.1.3):

$$\epsilon_r^o = \frac{1}{2f\sqrt{1+(\gamma)^2}} \frac{du_r^o}{d\gamma} - \frac{w}{2f\left[1+(\gamma)^2\right]^{3/2}} , \qquad 6.4.4$$

$$\epsilon_\theta^o = \frac{u_r^o}{2f\gamma\sqrt{1+(\gamma)^2}} - \frac{w}{2f\sqrt{1+(\gamma)^2}} . \qquad 6.4.5$$

The equations of equilibrium also become simplified (see equations 6.1.4, 6.1.5, 6.1.6):

$$\gamma \frac{dN_r}{d\gamma} + N_r - N_\theta + 2f\gamma\sqrt{1+(\gamma)^2} \quad p_r = 0 , \qquad 6.4.6$$

$$\frac{1}{1+(\gamma)^2} N_r + N_\theta + 2f\sqrt{1+(\gamma)^2} \quad p_n = 0. \qquad 6.4.7$$

The pertinent stress-strain relations (see 6.1.7, 6.1.8, 6.1.9)

are the following:

$$N_r = \frac{Eh}{1-\nu^2}\left(\epsilon_r^o + \nu\,\epsilon_\theta^o\right), \qquad 6.4.8$$

$$N_\theta = \frac{Eh}{1-\nu^2}\left(\epsilon_\theta^o + \nu\,\epsilon_r^o\right). \qquad 6.4.9$$

The loading which is of interest is caused by the dead weight of the shell. From section 6.3 we have

$$p_r = -\frac{\rho_0 h \Upsilon}{\sqrt{1+(\Upsilon)^2}}, \qquad 6.4.10$$

$$p_\theta = 0, \qquad 6.4.11$$

$$p_n = \frac{-\rho_0 h}{\sqrt{1+(\Upsilon)^2}}. \qquad 6.4.12$$

With the introduction of the loading terms (equations 6.4.10, 6.4.12) the two equilibrium equations can be combined into a single ordinary differential equation:

$$\frac{dN_r}{d\gamma} + N_r\left[\frac{1}{\gamma} + \frac{1}{\gamma[1+(\gamma)^2]}\right] = 2fh\rho_0\left(\gamma + \frac{1}{\gamma}\right) \qquad 6.4.13$$

The formal solution of the differential equation (6.4.13) can be expressed as

$$N_r = \frac{2f\rho_0 h}{3}\,\frac{\sqrt{1+(\gamma)^2}}{(\gamma)^2}\left\{\left[1+(\gamma)^2\right]^{3/2} + C_1\right\} \qquad 6.4.14$$

where C_1 is a constant to be determined by the boundary conditions.

Note that the differential equation (6.4.13) possesses a regular singularity at the apex, $\gamma = 0$. Thus the solution, as it is presented in equation 6.4.14, is valid only for $\gamma > 0$, i.e., for all $\gamma \geq \delta$ where $\delta > 0$ can be arbitrarily small. The case where the shell is closed at the apex will be discussed later. However, we may avoid the difficulties associated with the singularity by constructing the paraboloidal shell with a hole at the apex.

The force resultant N_θ is obtained by substituting

equation 6.4.14 into the equilibrium equation (6.4.7):

$$N_\theta(\gamma) = 2f\rho_0 h\left\{1 - \frac{1}{3(\gamma)^2\sqrt{1+(\gamma)^2}}\left[\left(1+(\gamma)^2\right)^{3/2} + C_1\right]\right\}. \qquad 6.4.15$$

We will require the displacements u_r^o, u_θ^o, and w not only because the deformations are of primary interest but also because all the boundary conditions may be prescribed in terms of them. By solving equations 6.4.8 and 6.4.9 for ϵ_r^o and ϵ_θ^o, and then introducing equations 6.4.14 and 6.4.15, the strains can be determined:

$$\epsilon_r^o = \frac{2f\rho_0}{3E}\left\{\frac{1+\nu}{(\gamma)^2} + 2(1-\nu) + (\gamma)^2 + C_1\left[\frac{(1+\nu)}{(\gamma)^2\sqrt{1+(\gamma)^2}} + \frac{1}{\sqrt{1+(\gamma)^2}}\right]\right\}, \qquad 6.4.16$$

$$\epsilon_\theta^o = \frac{2f\rho_0}{3E}\left\{2(1-\nu) - \frac{1+\nu}{(\gamma)^2} - \nu(\gamma)^2 - C_1\left[\frac{(1+\nu)}{(\gamma)^2\sqrt{1+(\gamma)^2}} + \frac{1}{\sqrt{1+(\gamma)^2}}\right]\right\} \qquad 6.4.17$$

The strain-displacement relations (6.4.6) can be rearranged to read

$$w = \frac{u_r^o}{\gamma} - 2f\sqrt{1+(\gamma)^2}\,\epsilon_\theta^o \qquad 6.4.18$$

and substituted into the other strain-displacement relation (6.4.4) to yield the differential equation for u_r° .

$$\frac{du_r^0}{d\gamma} - \frac{u_r^0}{\gamma[1+(\gamma)^2]} = 2f\sqrt{1+(\gamma)^2}\left(\epsilon_r^0 - \frac{\epsilon_\theta^0}{1+(\gamma)^2}\right). \qquad 6.4.19$$

If the shell is not closed at the apex, the solution to the differential equation (6.4.19) is

$$\begin{aligned} u_r^0(\gamma) = \frac{4f^2\rho_0}{3E\sqrt{1+(\gamma)^2}}\Big\{&(1+\nu)\gamma\,\ell n\,\gamma - \frac{1+\nu}{\gamma} + \left(\frac{3-\nu}{2}\right)(\gamma)^3 + \frac{1}{4}(\gamma)^5 \\ &+ C_1\Big[-(1+\nu)\gamma\,\ell n\left(\frac{1+\sqrt{1+(\gamma)^2}}{\gamma}\right) - \frac{(1+\nu)\sqrt{1+(\gamma)^2}}{\gamma} \\ &+ \gamma\sqrt{1+(\gamma)^2}\Big] + C_2\gamma\Big\}. \end{aligned} \qquad 6.4.20$$

The introduction of 6.4.20 into 6.4.18 yields

$$\begin{aligned} w(\gamma) = \frac{4f^2\rho_0}{3E\sqrt{1+(\gamma)^2}}\Big\{&(1+\nu)\left(\ell n\,\gamma - \frac{1}{(\gamma)^2}\right) + \left(\frac{3-\nu}{2}\right)(\gamma)^2 + \frac{1}{4}(\gamma)^4 + \left(1+(\gamma)^2\right) \\ &\left[2(\nu-1) + \frac{1+\nu}{(\gamma)^2} + \nu(\gamma)^2\right] + C_1\Big[-(1+\nu)\,\ell n\left(\frac{1+\sqrt{1+(\gamma)^2}}{\gamma}\right) \\ &+ \sqrt{1+(\gamma)^2}\,(1+\nu)\Big] + C_2\Big\}. \end{aligned} \qquad 6.4.21$$

Several different combinations of boundary conditions will be considered and explicit evaluations as well as numerical calculations will be made. These will be designated as Case 6.4.1,

Case 6.4.2 and Case 6.4.3. The numerical results will be presented in terms of the non-dimensionalized quantities

$$N_r^* = \frac{N_r}{2f\rho_0 h} \qquad 6.4.22$$

$$N_\theta^* = \frac{N_\theta}{2f\rho_0 h} \qquad 6.4.23$$

$$u_r^* = \frac{u_r E}{4f^2\rho_0} \qquad 6.4.24$$

$$w^* = \frac{w E}{4f^2\rho_0} \qquad 6.4.25$$

The choice of parameters for non-dimensionalization is obvious from equations (6.4.14), (6.4.15), (6.4.20) and (6.4.21).

6.4.1 BOTH EDGES RESTRAINED IN THE TANGENTIAL DIRECTIONS

The boundary conditions for a shell with both edges restrained in the tangential directions are specified by

$$u_r^o(\gamma_1) = 0, \qquad 6.4.1.1$$

$$u_r^o(\gamma_2) = 0 \qquad 6.4.1.2$$

where γ_1 and γ_2 are the boundaries of the shell (see figure 6.2.1) and are chosen such that

$$\gamma_1 < \gamma_2 \qquad 6.4.1.3$$

The solution for C_1 and C_2 can be put into the form

$$C_1 = \frac{B(\gamma_1)\,\gamma_2 - B(\gamma_2)\,\gamma_1}{A(\gamma_1)\,\gamma_2 - A(\gamma_2)\,\gamma_1} \qquad 6.4.1.4$$

$$C_2 = \frac{B(\gamma_2)\,A(\gamma_1) - B(\gamma_1)\,A(\gamma_2)}{A(\gamma_1)\,\gamma_2 - A(\gamma_2)\,\gamma_1} \qquad 6.4.1.5$$

where

$$A(\gamma_i) = \gamma_i\sqrt{1+(\gamma_i)^2} - (1+\nu)\left[\gamma_i \ln\left(\frac{1+\sqrt{1+(\gamma_i)^2}}{\gamma_i}\right) + \frac{\sqrt{1+(\gamma_i)^2}}{\gamma_i}\right], \tag{6.4.1.6}$$

$$B(\gamma_i) = A_1\left(\frac{1}{\gamma_i} - \gamma_i \ln \gamma_i\right) - A_2(\gamma_i)^3 - A_3(\gamma_i)^5 . \tag{6.4.1.7}$$

The possibility that the determinant Δ,

$$\Delta = A(\gamma_1)\gamma_2 - A(\gamma_2)\gamma_1 \tag{6.4.1.8}$$

in equations (6.4.1.4) and (6.4.1.5) vanishes for some γ_1 and γ_2 should be investigated. Physically this is implausible as it would mean some sort of ins ability. Therefore, one suspects that as a function of γ_2, the transcendental equation

$$\Delta = 0 \tag{6.4.1.9}$$

for any (fixed) positive γ_1 has no positive roots which are larger than γ_1. It can indeed be verified that Δ is monotonic and non-zero for all positive $\gamma_2 > \gamma_1$.

There are six separate configurations which have been analyzed. The results are shown in Figures 6.4.1.1 through 6.4.1.6 which contain curves of N_r^*, N_Θ^* and w^* as well as tables of values. It can

be seen from an examination of the table of values that u_r^* , the non-dimensional meridional displacement, is an order of magnitude smaller than W^* and it is for this reason that u_r^* has not been plotted. The boundary conditions that both edges are restrained in the tangential direction means that the shell must deform in the manner shown in Figure 6.4.1.

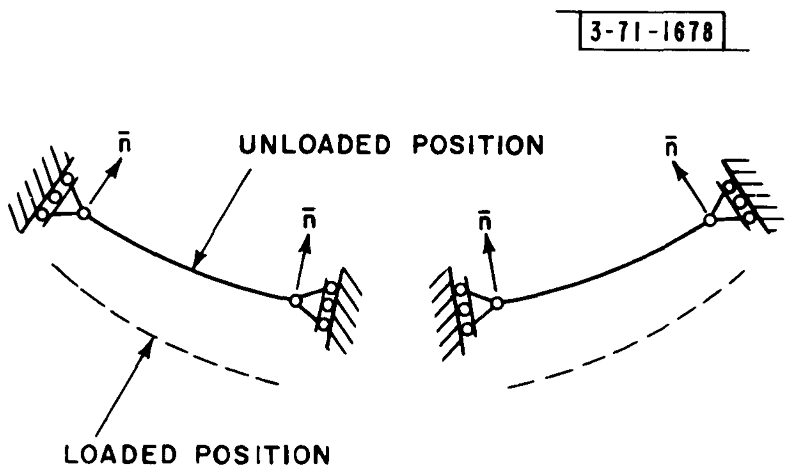


Figure 6.4.1

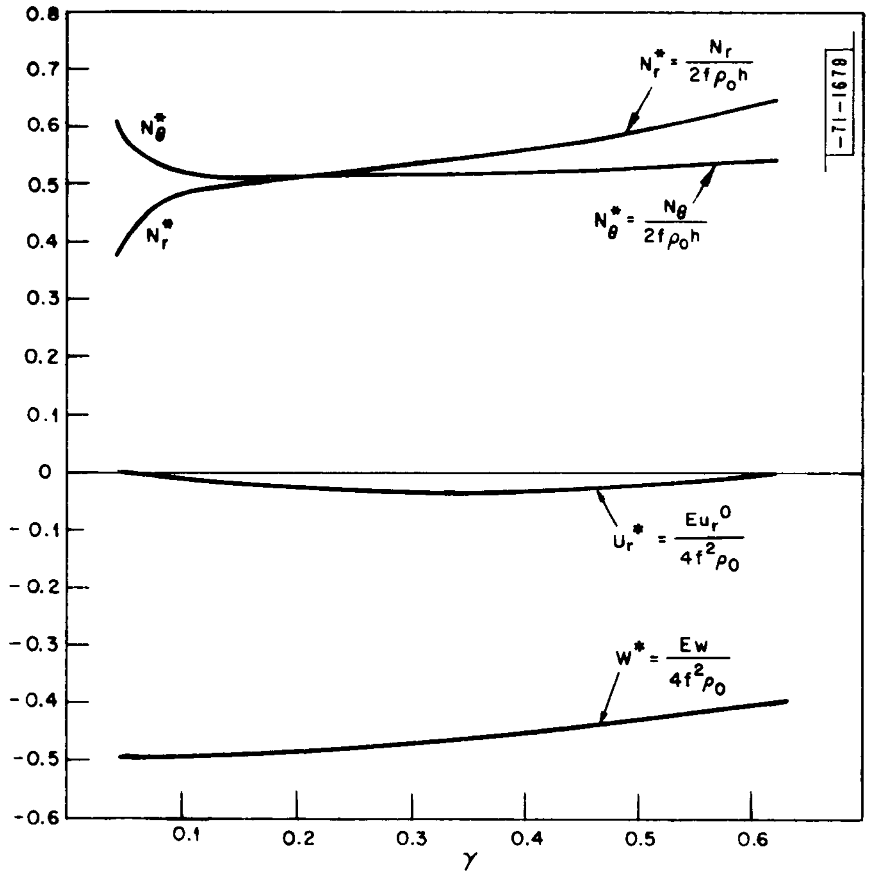
The restrained type of boundary conditions for membranes lead to distortions which at first glance seem peculiar. However, the boundary condition itself, i.e., the restraint of u_r^*, is somewhat unnatural because it would be very difficult to achieve in a practical situation. Schematically, a set of roller supports as shown in Figure

6.4.1 will prevent displacements along a tangent while freely permitting displacements along the normal to the surface of the shell.

A table which summarizes the cases studied in 6.4.1.1 follows:

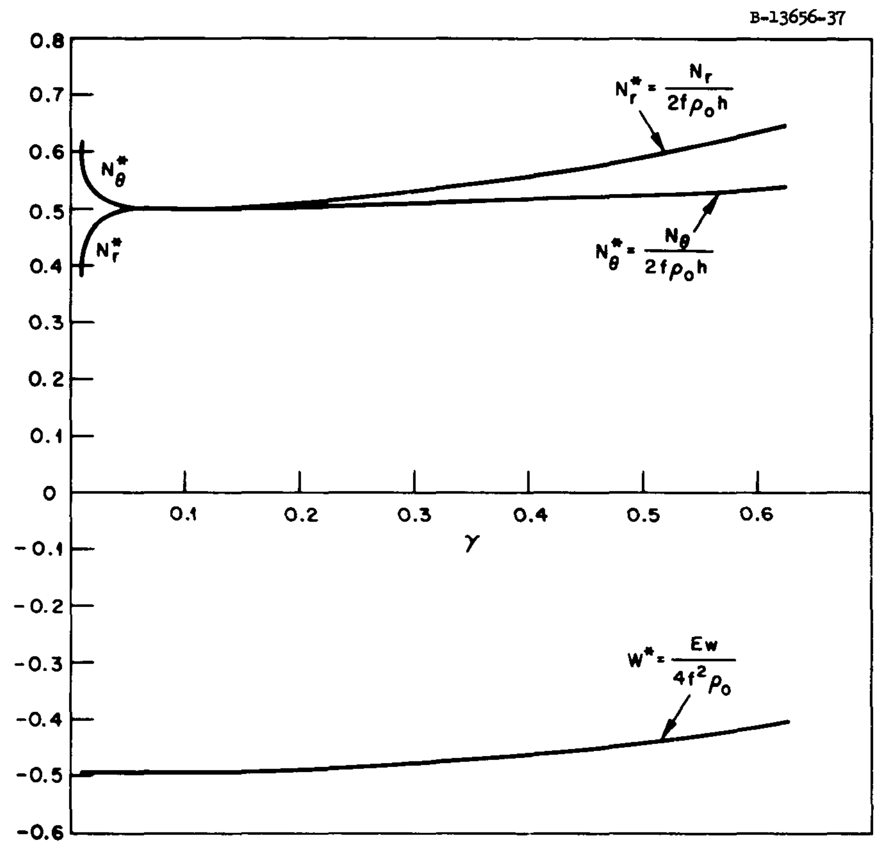
Table 6.4.1.1

Case No.	Restrained at γ_1	Restrained at γ_2
6.4.1.1	0.0446	0.6250
6.4.1.2	0.0104	0.6250
6.4.1.3	0.0104	0.3854
6.4.1.4	0.3854	0.6250
6.4.1.5	0.0446	0.8035
6.4.1.6	0.0446	1.0267



Case 6.4.1.1 Axi-Symmetric Gravity
Restrained $\gamma_1 = 0.0446$
Restrained $\gamma_2 = 0.625$

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0446	0.387	0.614	-0.000	-0.498
0.1091	0.485	0.520	-0.0130	-0.496
0.1736	0.504	0.511	-0.0220	-0.492
0.2380	0.517	0.511	-0.0287	-0.486
0.3025	0.532	0.513	-0.0329	-0.478
0.3670	0.548	0.517	-0.0341	-0.468
0.4315	0.568	0.521	-0.0319	-0.456
0.4960	0.590	0.526	-0.0258	-0.442
0.5605	0.615	0.532	-0.0153	-0.425
0.625	0.643	0.537	-0.000	-0.406



Case 6.4.1.2 Axi-Symmetric Gravity
Restrained γ_1 = 0.0104
Restrained γ_2 = 0.625

γ	N_r^*	N_θ^*	u_r^*	w^*
0.0104	.386	.614	-.000	-.498
0.0208	.472	.528	-.0023	-.498
0.0624	.498	.504	-.0089	-.487
0.1041	.503	.502	-.0148	-.496
0.1458	.507	.503	-.020	-.493
0.1874	.513	.505	-.0250	-.490
0.2291	.519	.506	-.0290	-.486
0.2708	.527	.509	-.032	-.482
0.3124	.536	.511	-.034	-.476
0.3541	.547	.514	-.0347	-.470
0.3958	.558	.517	-.0340	-.462
0.4374	.571	.521	-.6319	-.454
0.4791	.585	.524	-.0280	-.445
0.5208	.600	.528	-.022	-.435
0.5624	.617	.531	-.0150	-.424
0.6041	.634	.535	-.0055	-.412
0.625	.644	.537	-.000	-.405

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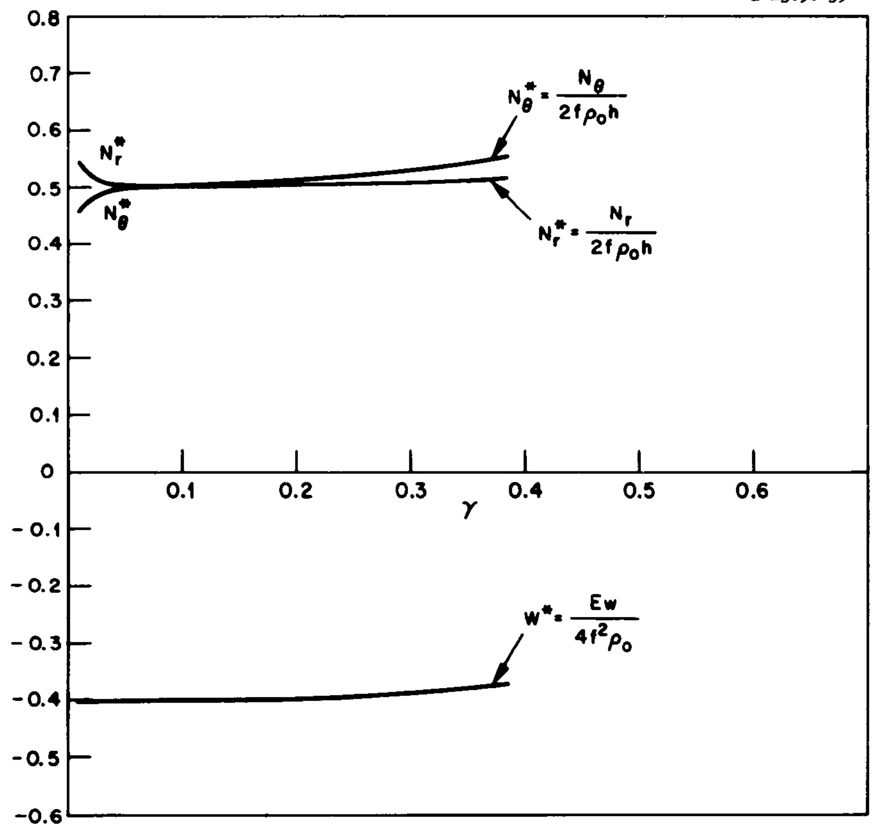
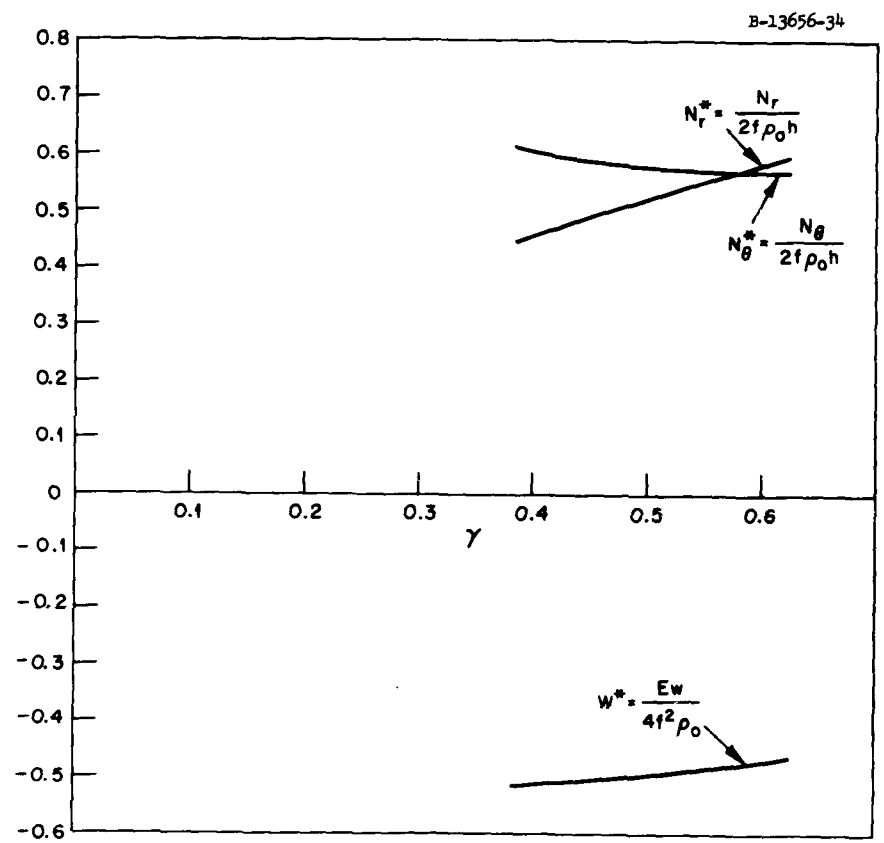


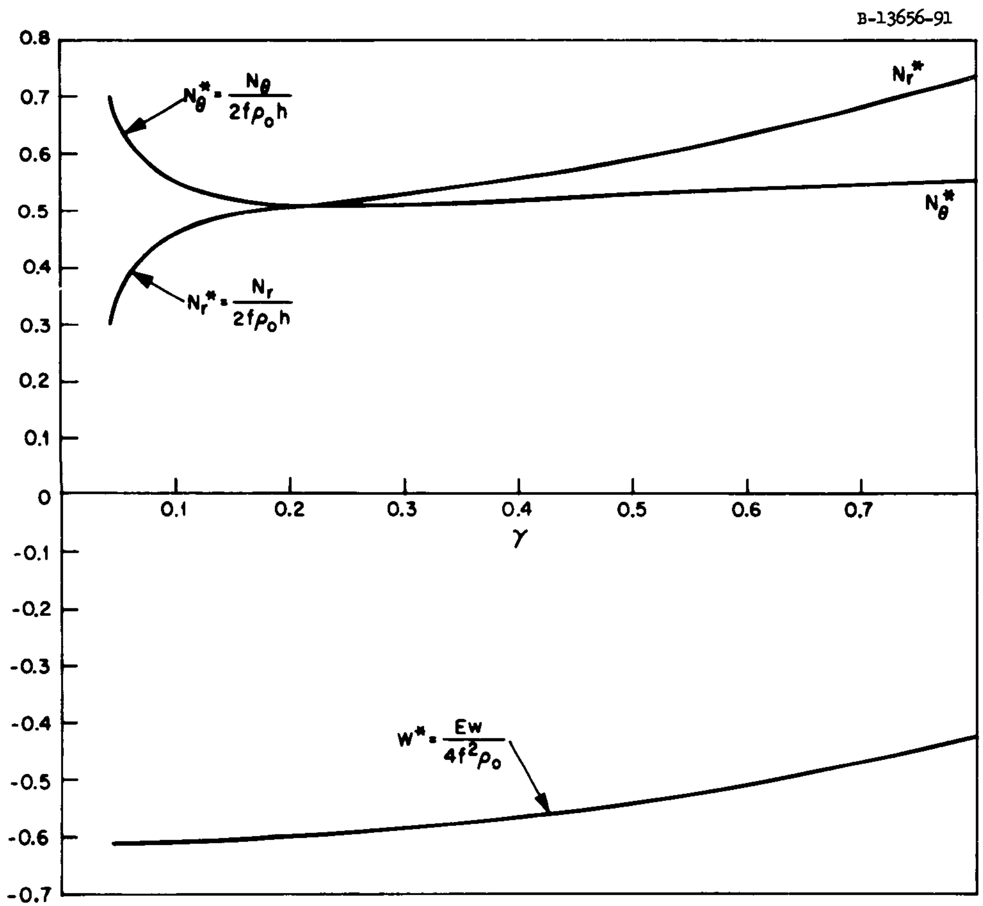
Case 6.4.1.3 Axi-Symmetric Gravity
Restrained $\gamma_1 = 0.0104$
Restrained $\gamma_2 = 0.3854$

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0104	.459	.540	.000	-.402
0.0358	.497	.503	-.0017	-.402
0.0612	.500	.502	-.00304	-.402
0.0866	.502	.502	-.0042	-.401
0.1121	.504	.502	-.005	-.400
0.1375	.507	.502	-.006	-.399
0.1693	.510	.504	-.007	-.328
0.1947	.514	.505	-.0075	-.396
0.2201	.518	.506	-.0077	-.394
0.2455	.522	.507	-.00756	-.392
0.2710	.527	.509	-.0071	-.389
0.2964	.532	.510	-.0062	-.387
0.3218	.538	.512	-.005	-.384
0.3472	.545	.514	-.0034	-.380
0.3727	.552	.516	-.0012	-.377
0.3854	.555	.516	.000	-.375



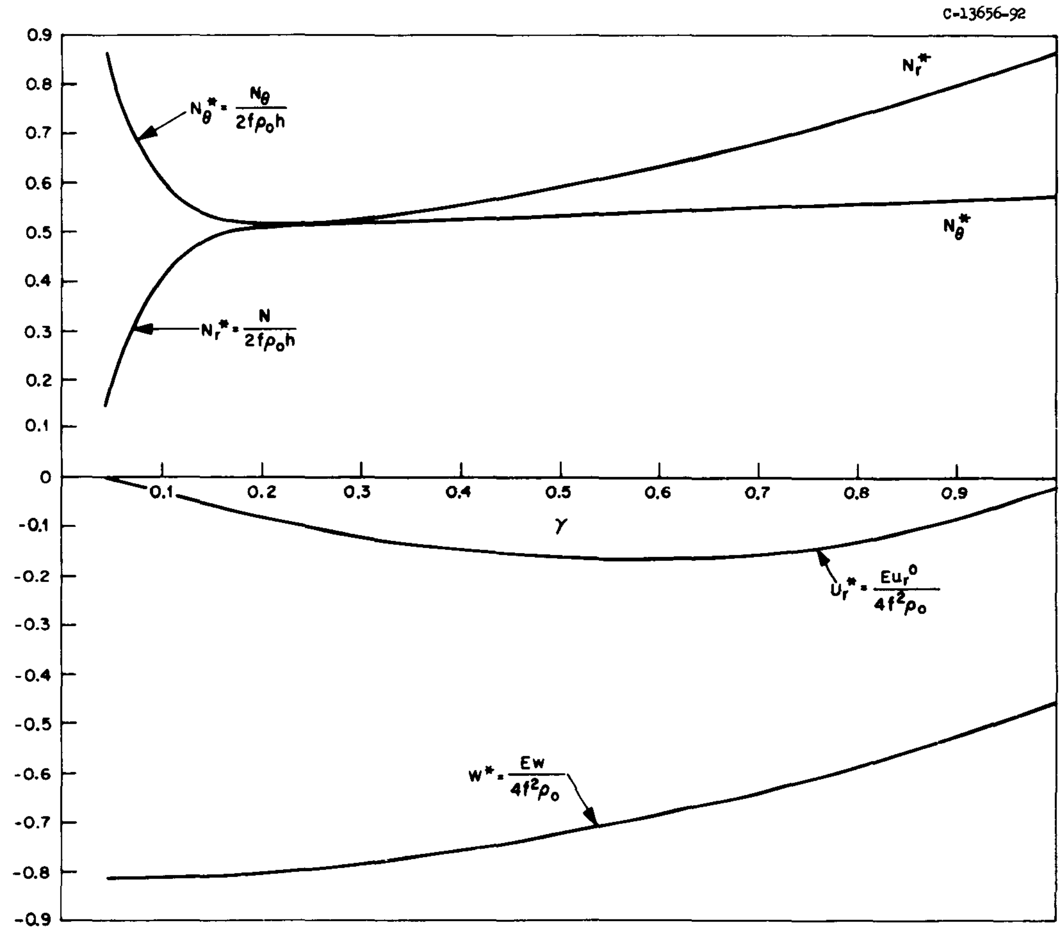
Case 6.4.1.4 Axi-Symmetric Gravity
Restrained γ_1 = 0.3854
Restrained γ_2 = 0.625

γ	N_r^*	N_θ^*	u_r^*	w^*
0.3854	.443	.614	-.000	-.516
0.4016	.456	.608	-.0026	-.514
0.4179	.468	.602	-.0048	-.511
0.4341	.480	.596	-.0066	-.508
0.4503	.491	.592	-.0080	-.505
0.4666	.502	.588	-.0091	-.502
0.4828	.512	.585	-.0098	-.499
0.4991	.522	.582	-.0101	-.495
0.5153	.532	.579	-.010	-.492
0.5316	.542	.577	-.0096	-.488
0.5478	.552	.576	-.0088	-.484
0.5640	.561	.574	-.0077	-.480
0.5803	.571	.573	-.006	-.475
0.5965	.580	.572	-.004	-.471
0.6128	.590	.571	-.0020	-.466
0.625	.597	.571	-.000	-.462



Case 6.4.1.5 Axi-Symmetric Gravity
Restrained γ_1 = 0.0446
Restrained γ_2 = 0.8035

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0446	.300	.700	-0.000	-0.610
0.1339	.484	.524	-0.030	-0.606
0.2232	.510	.513	-0.050	-0.598
0.3124	.532	.515	-0.066	-0.584
0.4017	.557	.520	-0.075	-0.567
0.4910	.587	.526	-0.075	-0.544
0.5803	.622	.533	-0.067	-0.517
0.6696	.663	.541	-0.049	-0.485
0.7589	.709	.549	-0.0194	-0.448
0.8035	.735	.553	-0.000	-0.426



Case 6.4.1.6 Axi-Symmetric Gravity
Restrained γ_2 = 0.0446
Restrained γ_2 = 1.0267

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0446	.1418	.8584	.000	-0.8166
0.1339	.4665	.5416	-0.0544	-0.8116
0.2232	.5039	.5199	-0.0942	-0.8000
0.3124	.5287	.5182	-0.1267	-0.7825
0.4017	.5552	.5219	-0.1512	-0.7594
0.4910	.5860	.5277	-0.1664	-0.7309
0.5803	.6218	.5347	-0.1709	-0.6972
0.6696	.6629	.5422	-0.1639	-0.6581
0.7589	.7093	.5499	-0.1442	-0.6133
0.8482	.7609	.5574	-0.1110	-0.5625
0.9374	.8179	.5646	-0.0633	-0.5052
1.0267	.8801	.5715	-0.000	-0.4407

6.4.2 ONE EDGE FREE AND ONE EDGE RESTRAINED IN THE TANGENTIAL DIRECTION

The boundary conditions for a shell with one edge free and the other edge restrained in the tangential directions are specified by

$$N_r(r_1) = 0, \qquad 6.4.2.1$$

$$u_r^o(r_2) = 0 \qquad 6.4.2.2$$

where r_1 is the coordinate of the free edge and r_2 is the edge which is restrained. The free edge boundary condition, ecuation 6.4.2.1, determines the constant C_1 (see equation 6.4.14):

$$C_1 = -\left[1 + (r_1)^2\right]^{3/2}. \qquad 6.4.2.3$$

The restraint condition, equation 6.4.2.2, enables us to determine the remaining constant (see equation 6.4.20):

$$C_2 = \left\{ (1+\nu)\left[\frac{\sqrt{1+(\gamma_2)^2}}{(\gamma_2)^2} + \ln\left(\frac{1+\sqrt{1+(\gamma_2)^2}}{\gamma_2}\right)\right] - \sqrt{1+(\gamma_2)^2} \right\} C_1$$
$$- (1+\nu)\ln \gamma_2 + \frac{(1+\nu)}{(\gamma_2)^2} - \frac{(3-\nu)}{2}(\gamma_2)^2 - \frac{1}{4}(\gamma_2)^4 . \qquad 6.4.2.4$$

A total of twenty-nine separate cases have been solved. These represent various combinations of shells with the inner circular boundary unsupported and the outer circular boundary restrained against tangential motion. Additionally, there are cases in which the shell is supported at the inner boundary and is left free at the outer boundary.

The value of N_θ^* at the free boundary is seen always to be

$$N_\theta^* = 1 \qquad 6.4.2.5$$

by examining the equilibrium condition, equation 6.4.7, and the loading term, equation 6.4.12.

Again, it should be noted that the restrained type of boundary condition will sometimes yield nonsensical results. Thus, case 6.4.2.14, in which the shell is supported at an inner radius of only γ = .0104, shows non-dimensionalized displacements of over 2000. Obviously, the answer, while mathematically correct, is physically unrealizable. The case has been included as a mathematically interesting example of the effect of a load concentration.

A table which summarizes the cases studied in this section follows:

Table 6.4.2.1

Case No.	Free at γ_1	Restrained at γ_2
6.4.2.1	.0104	.1041
6.4.2.2	.0104	.0520
6.4.2.3	.0104	.2083
6.4.2.4	.0104	.3854
6.4.2.5	.0104	.4687
6.4.2.6	.0446	.4464
6.4.2.7	.0104	.6250
6.4.2.8	.0223	.6250
6.4.2.9	.0446	.6250
6.4.2.10	.0892	.6250
6.4.2.11	.1785	.6250
6.4.2.12	.1339	.6250
6.4.2.13	.4017	.6250
6.4.2.14	.6250	.0104
6.4.2.15	.6250	.0520
6.4.2.16	.6250	.2083
6.4.2.17	.6250	.3854
6.4.2.18	.6250	.4687
6.4.2.19	.0104	.9990
6.4.2.20	.1785	.8035
6.4.2.21	.4017	.8035
6.4.2.22	.1785	1.0267
6.4.2.23	.4017	1.0267
6.4.2.24	.8020	.2083
6.4.2.25	.8020	.3854
6.4.2.26	.8020	.4687
6.4.2.27	.9990	.3854
6.4.2.28	1.0000	.4687
6.4.2.29	.0104	.8020

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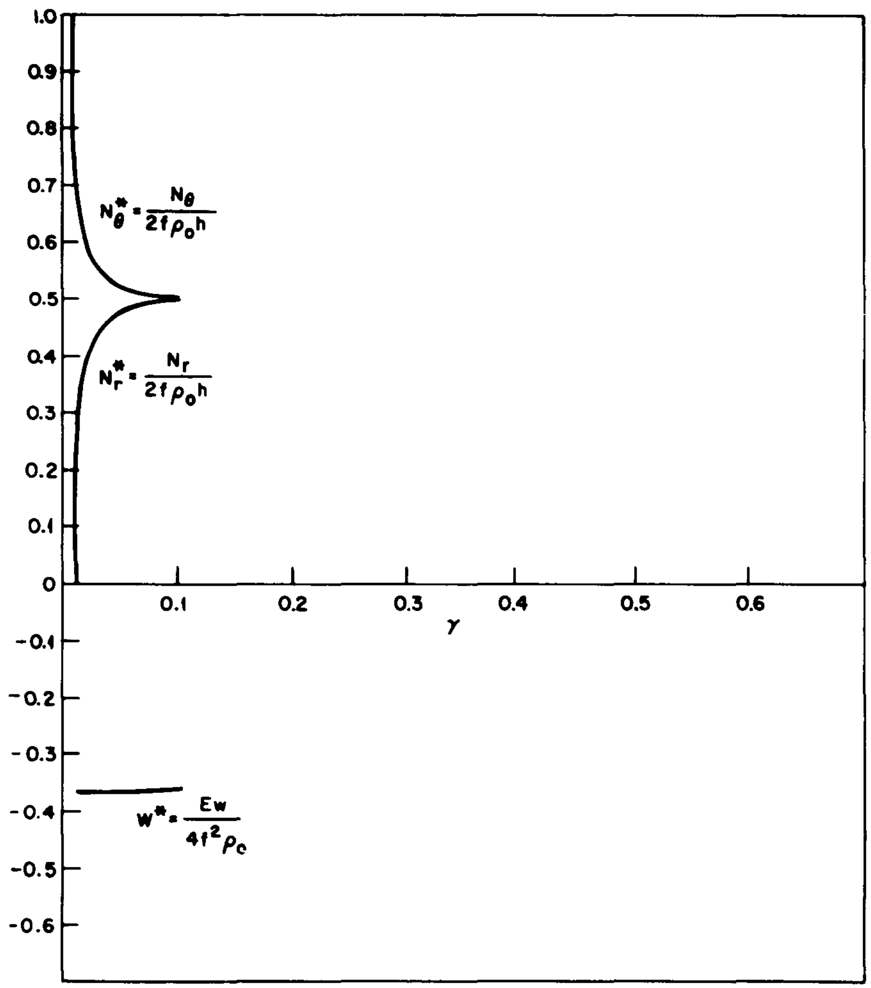
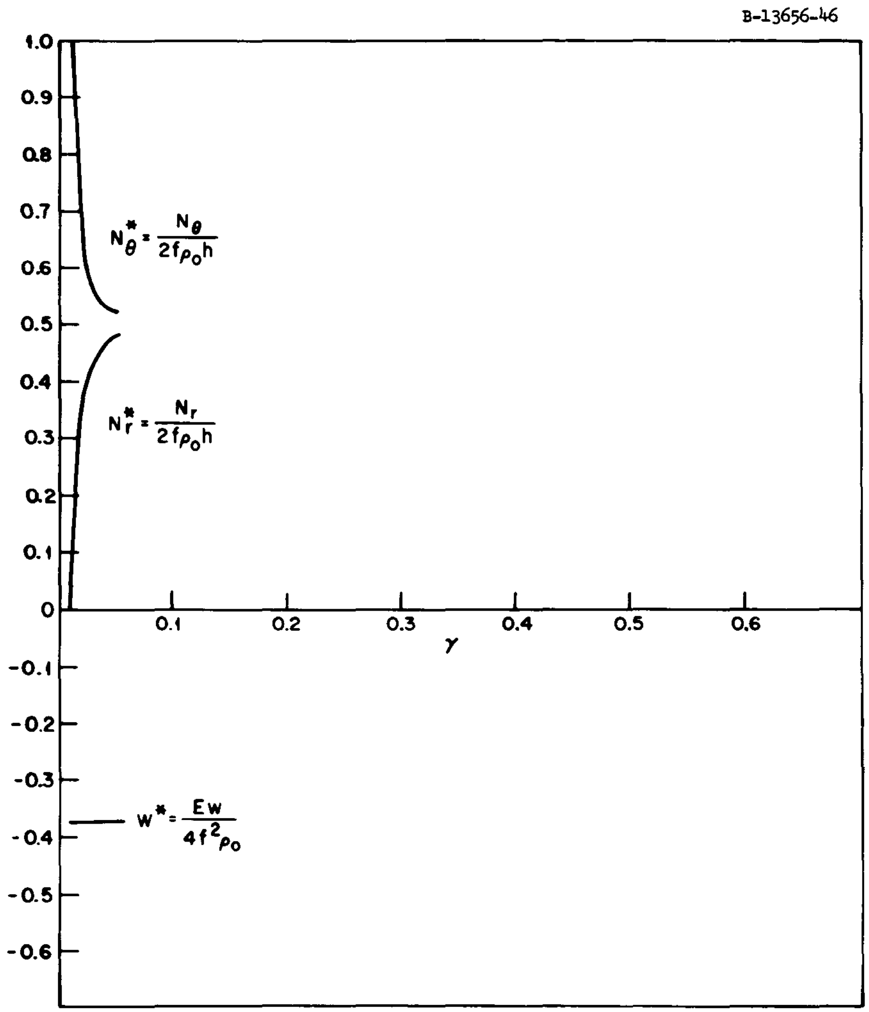


Case 6.4.2.1 Axi-Symmetric Gravity
Free γ_1 = 0.0104
Restrained γ_2 = 0.1041

γ	N_r^*	N_θ^*	u_r^*	w^*
0.0104	.000	1.000	.00666	-.360
0.0312	.445	.556	.00195	-.360
0.0520	.481	.520	.00087	-.360
0.0833	.494	.509	.00014	-.359
0.1041	.499	.506	.000	-.358



Case 6.4.2.2 Axi-Symmetric Gravity
Free γ_1 = 0.0104
Restrained γ_2 = 0.0520

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0104	.000	1.00	.00649	-.377
0.0312	.445	.556	.0014	-.377
0.0520	.481	.520	.000	-.376

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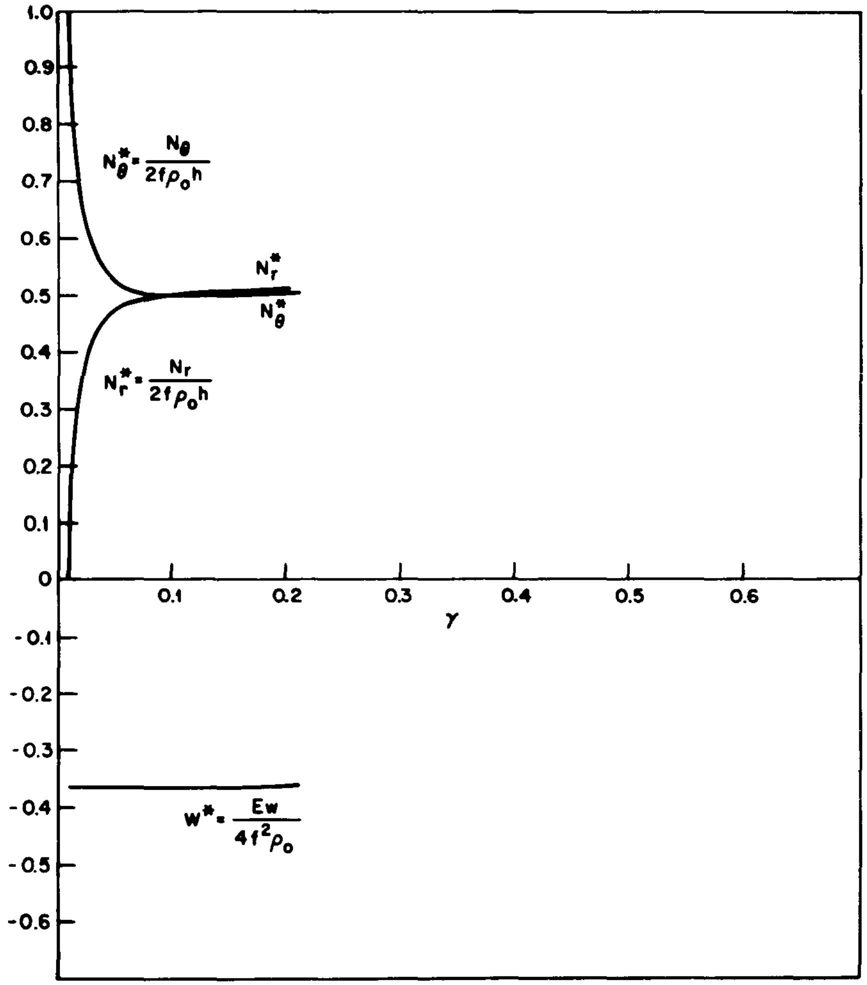

Case 6.4.2.3 Axi-Symmetric
Free $\gamma_1 = 0.0104$
Restrained $\gamma_2 = 0.2083$

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0104	.000	1.00	.0066	-.366
0.0520	.481	.520	.0	-.366
0.0937	.497	.507	-.0	-.365
0.1354	.504	.505	-.0008	-.364
0.1770	.510	.506	-.000	-.362
0.2083	.515	.506	-.000	-.360

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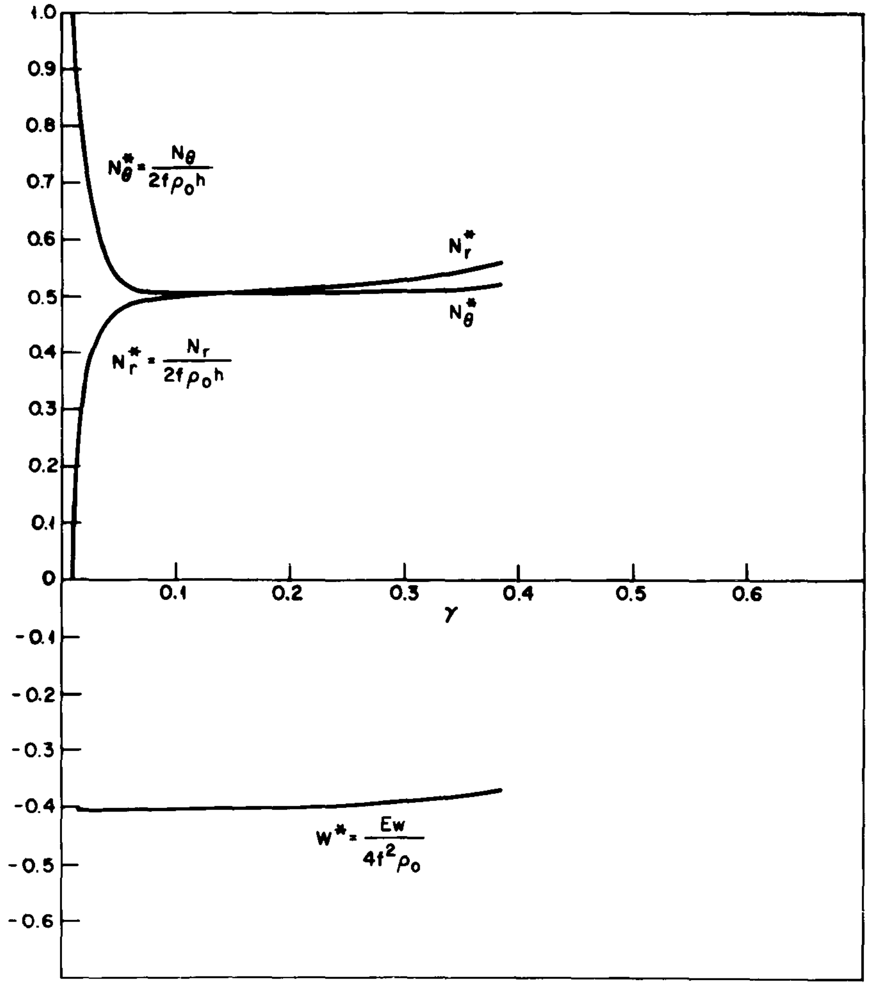

Case 6.4.2.4 Axi-Symmetric Gravity
Free γ_1 = 0.0104
Restrained γ_2 = 0.3854

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0104	.000	1.00	.0062	-.403
0.0520	.481	.520	-.0014	-.402
0.0937	.497	.507	-.0039	-.401
0.1354	.504	.505	-.0058	-.400
0.1770	.510	.506	-.0070	-.397
0.2187	.517	.507	-.0075	-.394
0.2604	.525	.509	-.0072	-.391
0.3020	.533	.511	-.0059	-.386
0.3437	.544	.514	-.0036	-.381
0.3854	.555	.517	.000	-.376

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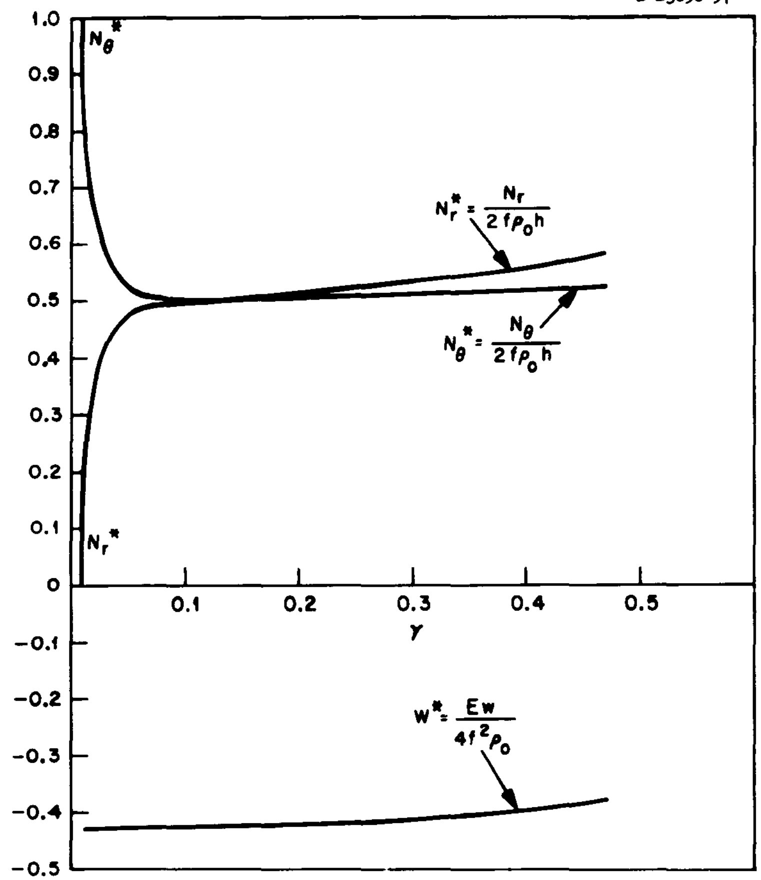
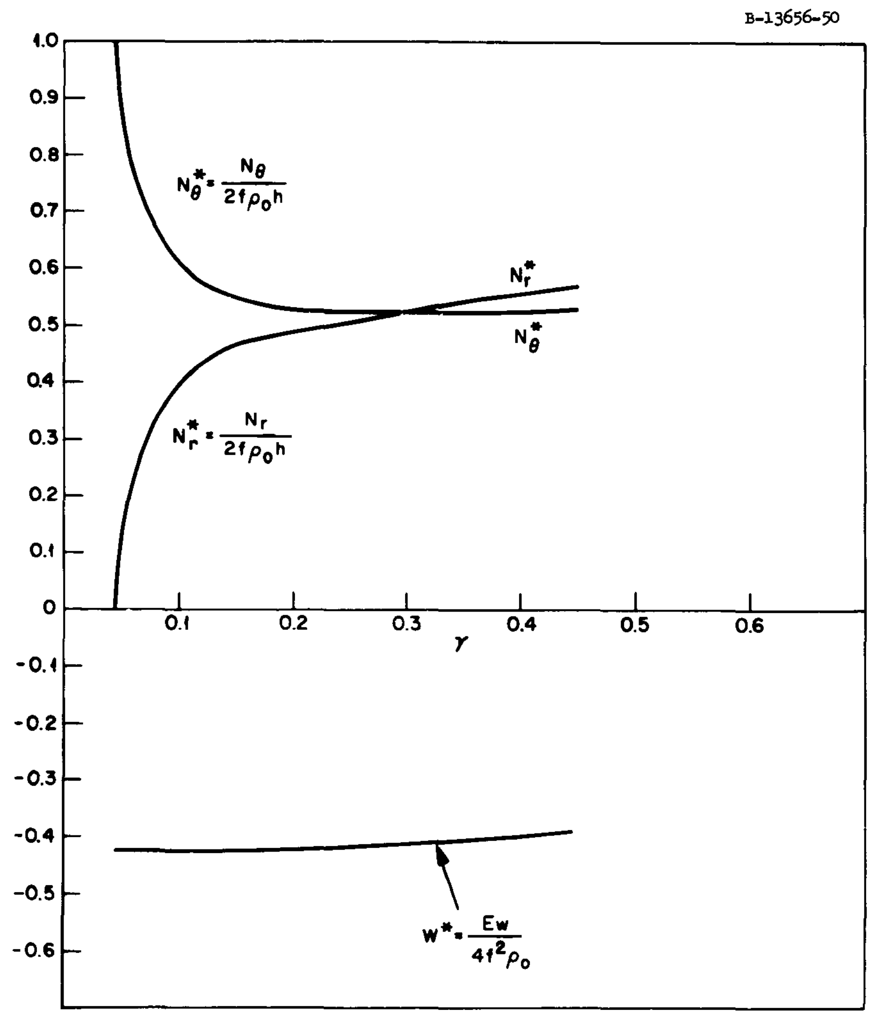


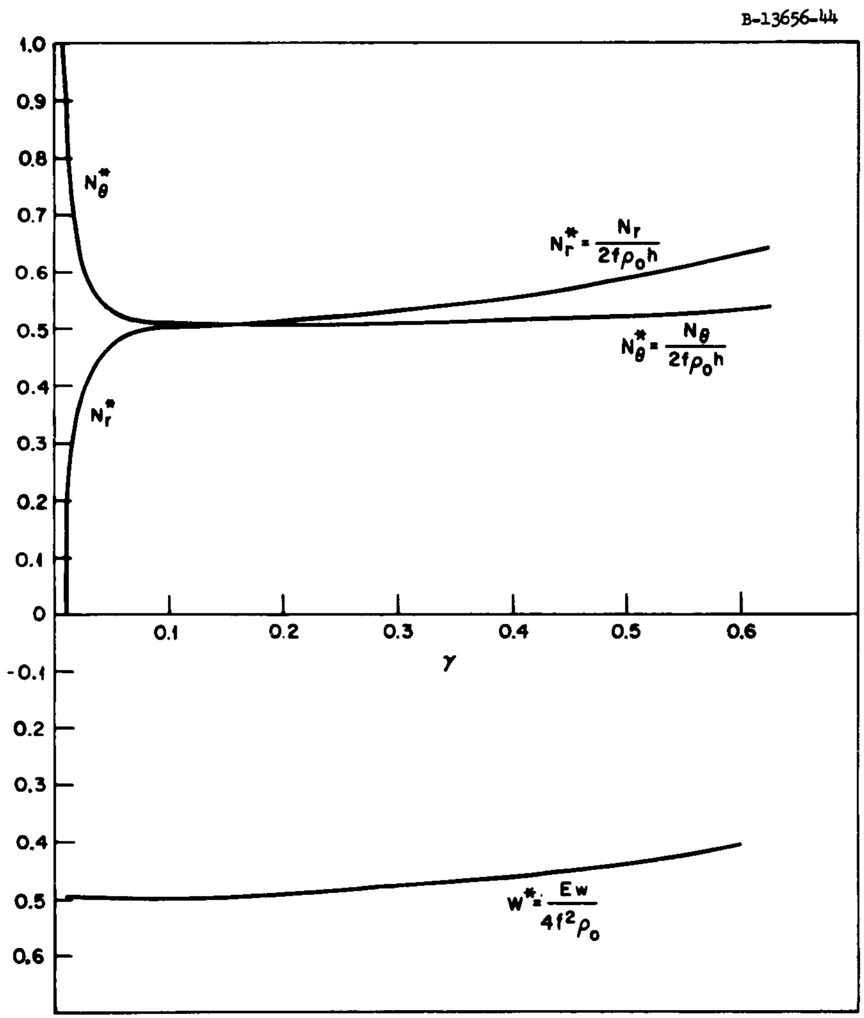
Case 6.4.2.5 Axi-Symmetric Gravity
Free γ_1 = 0.0104
Restrained γ_2 = 0.4687

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0104	.000	1.00	.0059	-.429
0.0520	.481	520	-.0027	-.429
0.0937	.497	.507	-.0064	-.428
0.1354	.504	.505	-.0093	-.426
0.1770	.510	.506	-.0116	-.424
0.2187	.517	.507	-.0132	-.420
0.2604	.525	.509	-.0139	-.417
0.3020	.533	.511	-.0136	-.412
0.3437	.544	.514	-.0122	-.407
0.3854	.555	.517	-.00960	-.400
0.4270	.567	.520	-.0055	-.393
0.4687	.581	.523	.000	-.386



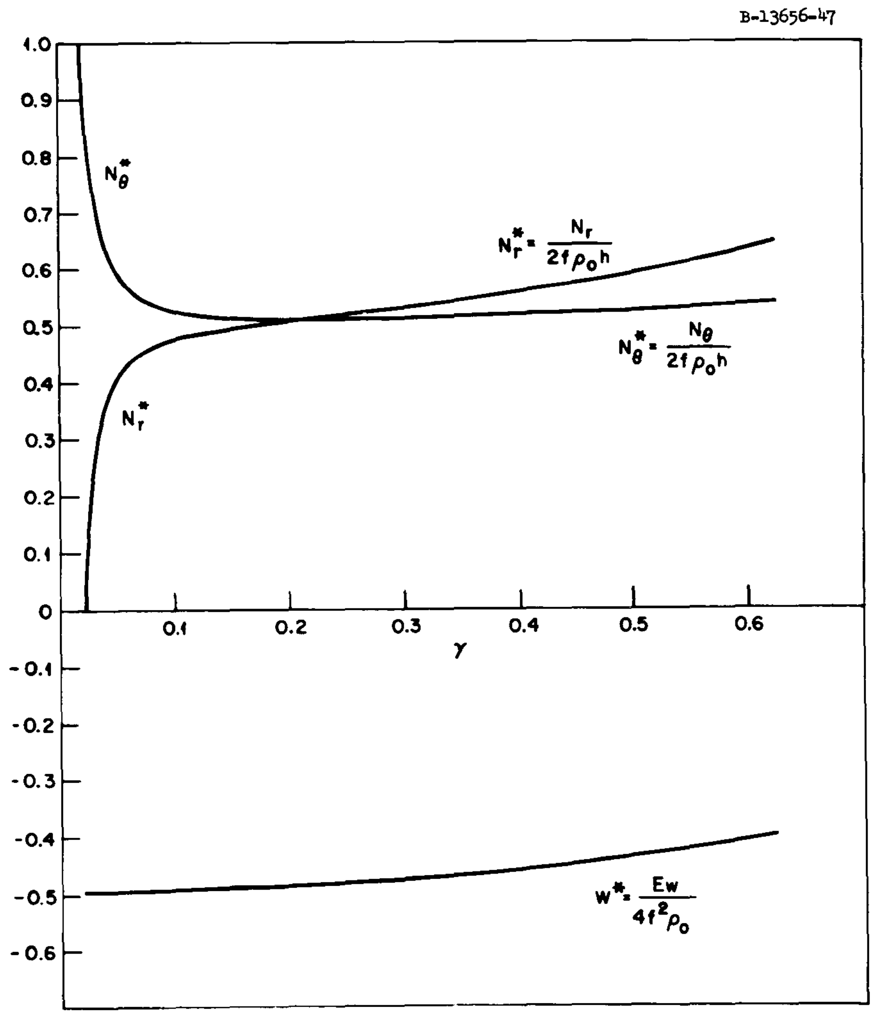
Case 6.4.2.6 Axi-Symmetric Gravity
Free γ_1 = 0.0446
Restrained γ_2 = 0.4464

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0446	.000	1.00	.0257	-.426
0.0613	.236	.764	.0164	-.427
0.1283	.445	.562	.0009	-.424
0.1953	.488	.530	-.00574	-.421
0.2622	.511	.522	-.00882	-.415
0.3292	.531	.521	-.00870	-.408
0.3962	.552	.523	-.00515	-.398
0.4464	.568	.526	.000	-.389



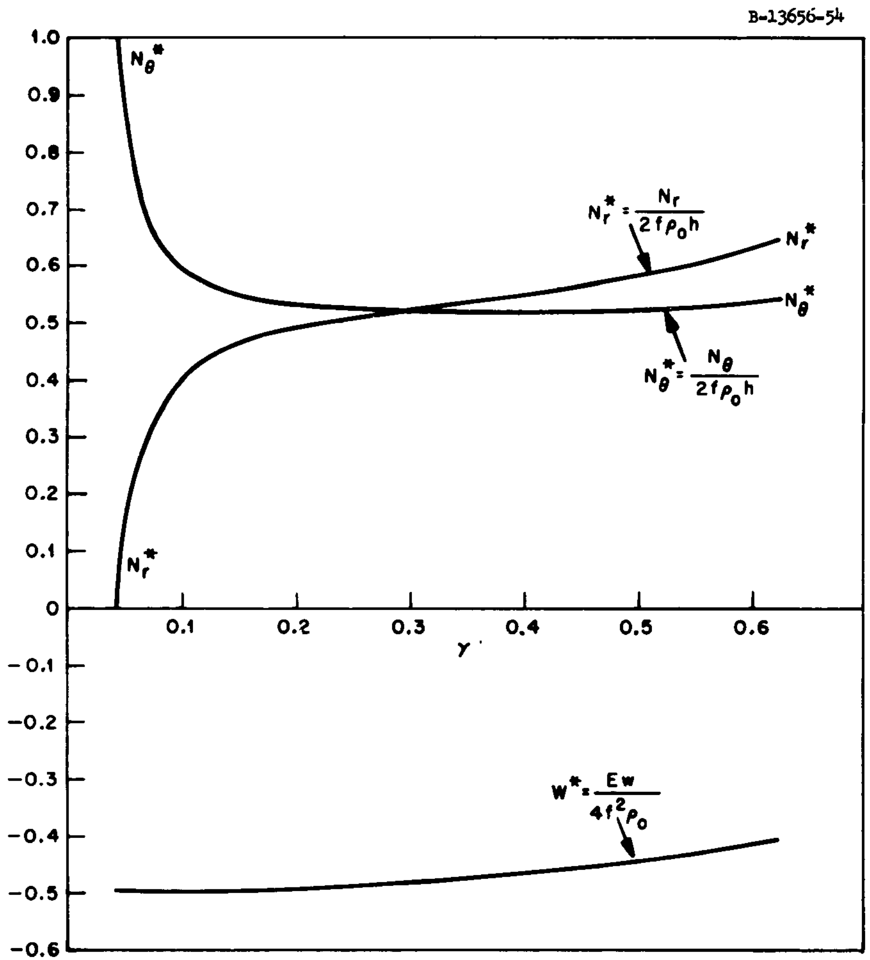
Case 6.4.2.7 Axi-Symmetric Gravity
Free $\gamma_1 = 0.0104$
Restrained $\gamma_2 = 0.625$

γ	N_r^*	N_θ^*	u_r^*	w^*
0.0104	0.	1.000	.00522	-.498
0.0520	.481	.520	-.00630	-.498
0.1145	.501	.506	-.0157	-.495
0.1770	.510	.506	-.0236	-.491
0.2395	.520	.508	-.0296	-.485
0.3020	.533	.511	-.0335	-.478
0.3645	.549	.515	-.0345	-.468
0.4270	.567	.520	-.0325	-.456
0.4895	.589	.525	-.0268	-.443
0.5520	.612	.531	-.0170	-.427
0.6145	.639	.536	-.00282	-.409
0.625	.644	.537	-.000	-.406



Case 6.4.2.8 Axi-Symmetric Gravity
Free $\gamma_1 = 0.0223$
Restrained $\gamma_2 = 0.625$

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0223	.000	1.00	.0112	-.498
0.0330	.273	.728	.00490	-.498
0.0761	.459	.544	-.00686	-.497
0.1191	.487	.519	-.0143	-.495
0.1622	.500	.512	-.0204	-.493
0.2052	.510	.511	-.0254	-.489
0.2483	.519	.511	-.0294	-.485
0.2913	.529	.513	-.0323	-.480
0.3344	.539	.515	-.0338	-.474
0.3774	.551	.518	-.0340	-.466
0.4205	.564	.520	-.0325	-.458
0.4635	.578	.524	-.0294	-.449
0.5066	.594	.527	-.0243	-.439
0.5496	.611	.531	-.0174	-.428
0.5927	.629	.535	-.00827	-.416
0.625	.643	.537	-.000	-.406



Case 6.4.2.9 Axi-Symmetric Gravity
Free γ_1 = 0.0446
Restrained γ_2 = 0.625

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0446	0.	-1.0	.0224	-.498
0.0661	.273	-.728	.00986	-.498
0.1091	.420	-.585	-.00392	-.497
0.1736	.478	-.536	-.0165	-.494
0.2380	.503	-.524	-.0250	-.488
0.3025	.523	-.521	-.0302	-.480
0.3670	.542	-.522	-.0322	-.471
0.4315	-.563	-.525	-.0306	-.459
0.4960	.587	-.524	-.0250	-.445
0.5606	.612	-.533	-.0149	-.428
0.625	.641	-.539	-.000	-.409

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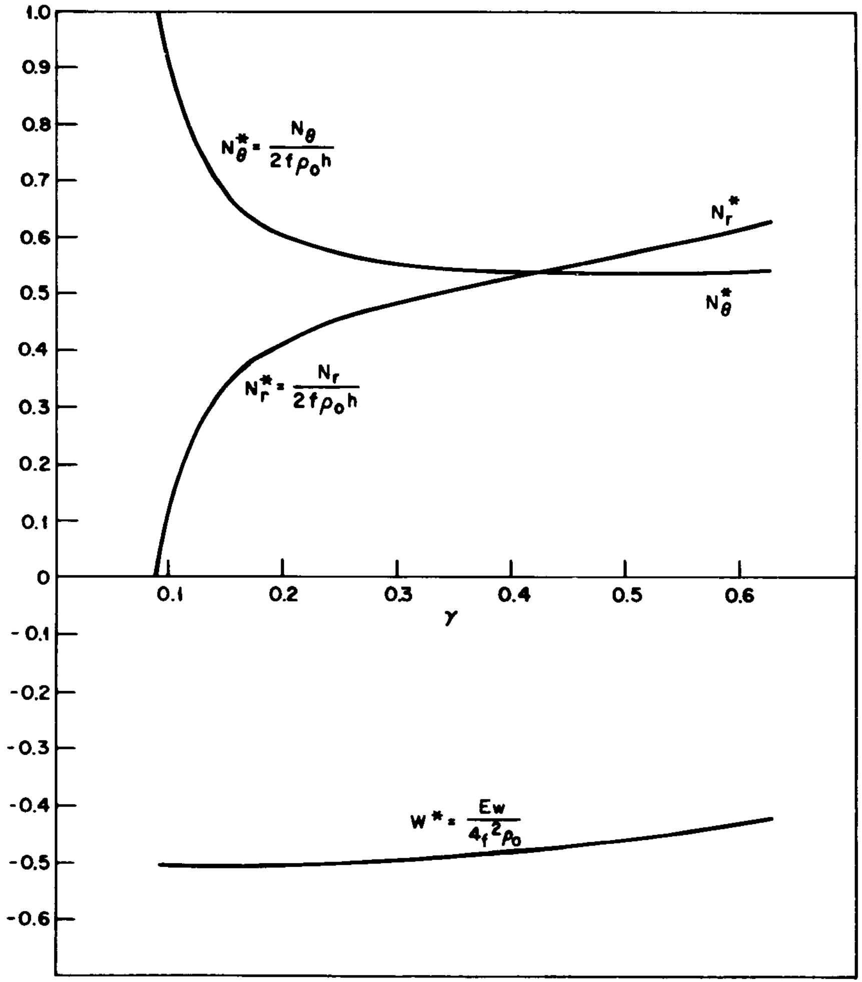

Case 6 4.2.10 Axi-Symmetric Gravity
Free γ_1 = 0.0892
Restrained γ_2 = 0.625

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0892	.000	1.000	.0447	-.503
0.1098	.172	.830	.0308	-.503
0.1923	.404	.610	-.000343	-.500
0.2747	.473	.560	-.0164	-.493
0.3571	.514	.544	-.0248	-.483
0.4395	.549	.540	-.0254	-.467
0.5219	.584	.541	-.0197	-.449
0.625	.632	.546	.000	-.420

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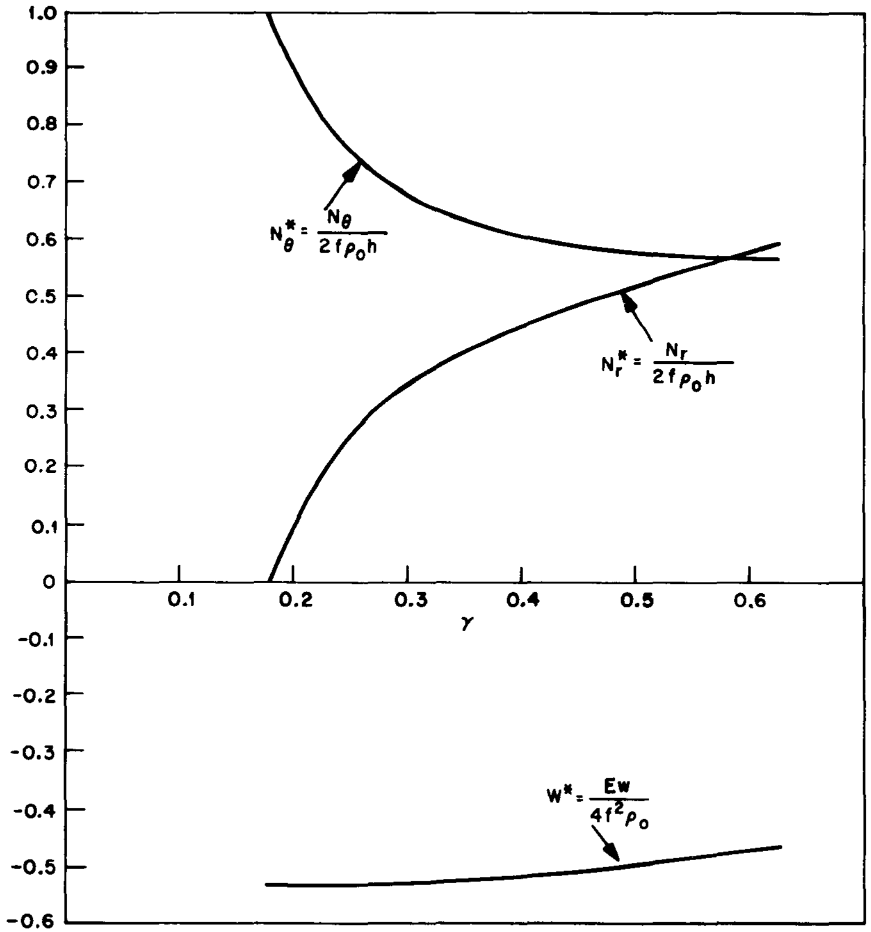

Case 6.4.2.II Axi-Symmetric Gravity
Free $\gamma_1 = 0.1785$
Restrained $\gamma_2 = 0.625$

γ	N_r^*	N_θ^*	u_r^*	w^*
0.1785	.000	1.000	.0866	-.531
0.1971	.1093	.910	.0725	-.531
0.2715	.302	.719	.0330	-.530
0.3459	.402	.640	.0093	-.523
0.4203	.467	.603	-.00427	-.512
0.4947	.517	.584	-.0095	-.498
0.5691	.562	.575	-.0070	-.480
0.625	.595	.572	.000	-.464

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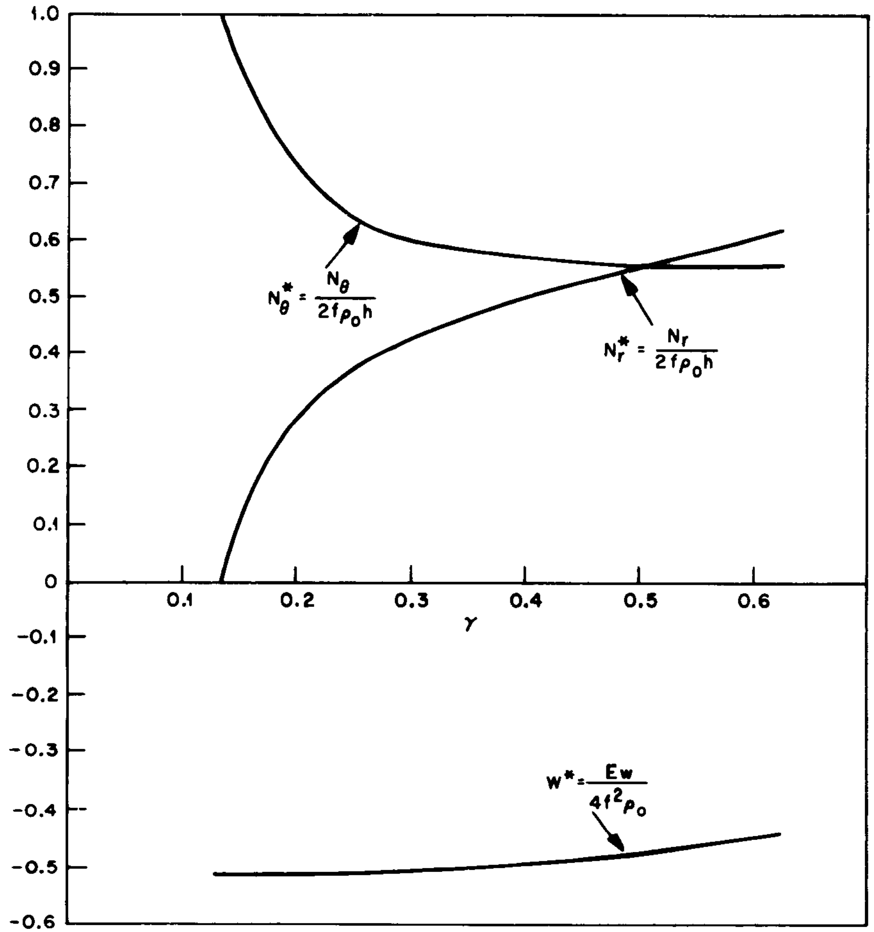

Case 6.4.2.12 Axi-Symmetric Gravity
Free γ_1 = 0.1339
Restrained γ_2 = 0.6249

γ	N_r^*	N_θ^*	U_r^*	w^*
0.1339	.000	1.00	.0664	-.513
0.1535	.122	.880	.0522	-.512
0.2321	.348	.669	.0159	-.510
0.3107	.438	.600	-.0042	-.500
0.3892	.492	.572	-.0149	-.494
0.4678	.536	.560	-.0177	-.480
0.5464	.576	.556	-.0128	-.461
0.625	.617	.556	.000	-.438

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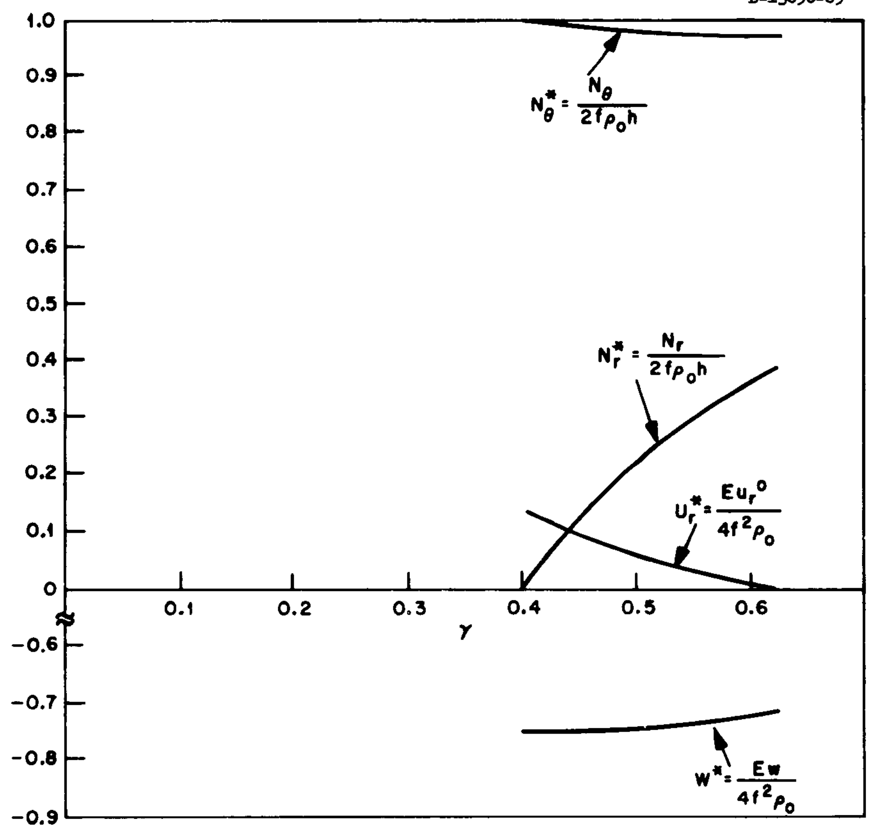

Case 6.4.2.13 Axi-Symmetric Gravity
Free γ_1 = 0.4017
Restrained γ_2 = 0.625

γ	N_r^*	N_θ^*	U_r^*	w^*
0.4017	.000	1.00	.134	-.744
0.4135	.0327	.972	.123	-.744
0.4605	.143	.882	.0841	-.741
0.5075	.230	.817	.0526	-.735
0.5545	.302	.769	.0273	-.727
0.6015	.363	.733	.0077	-.716
0.625	.391	.719	.000	-.710

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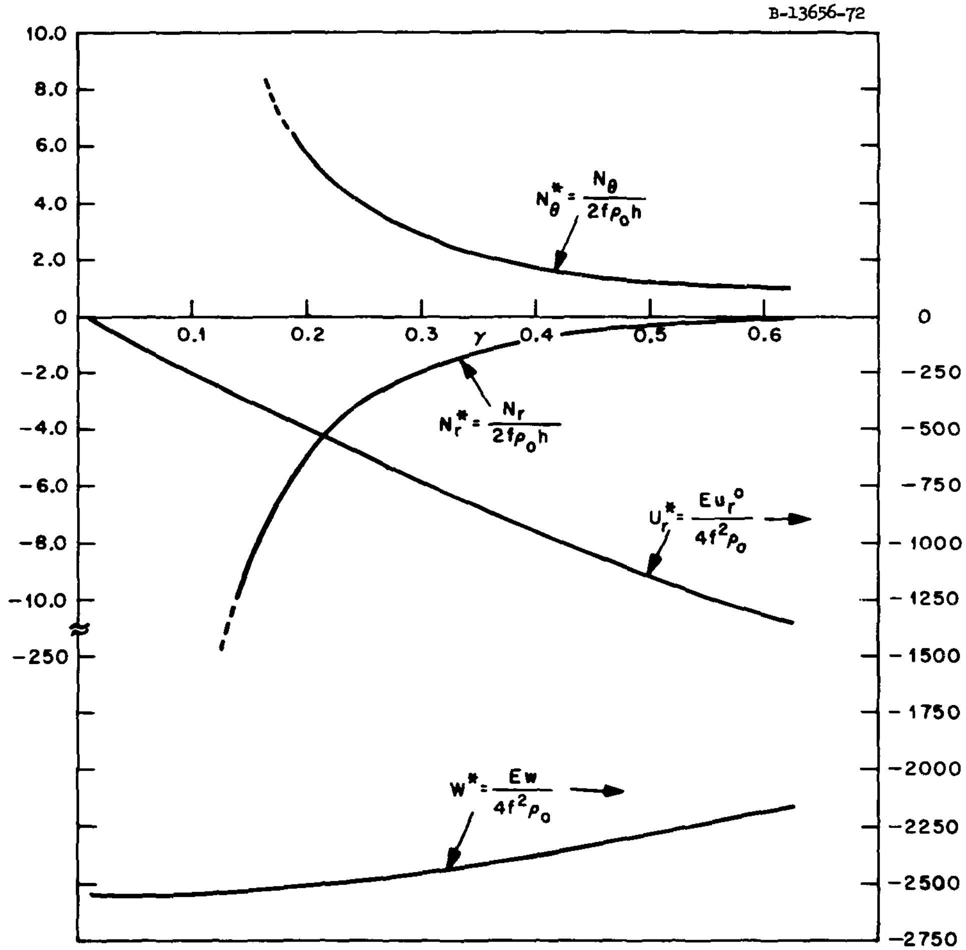
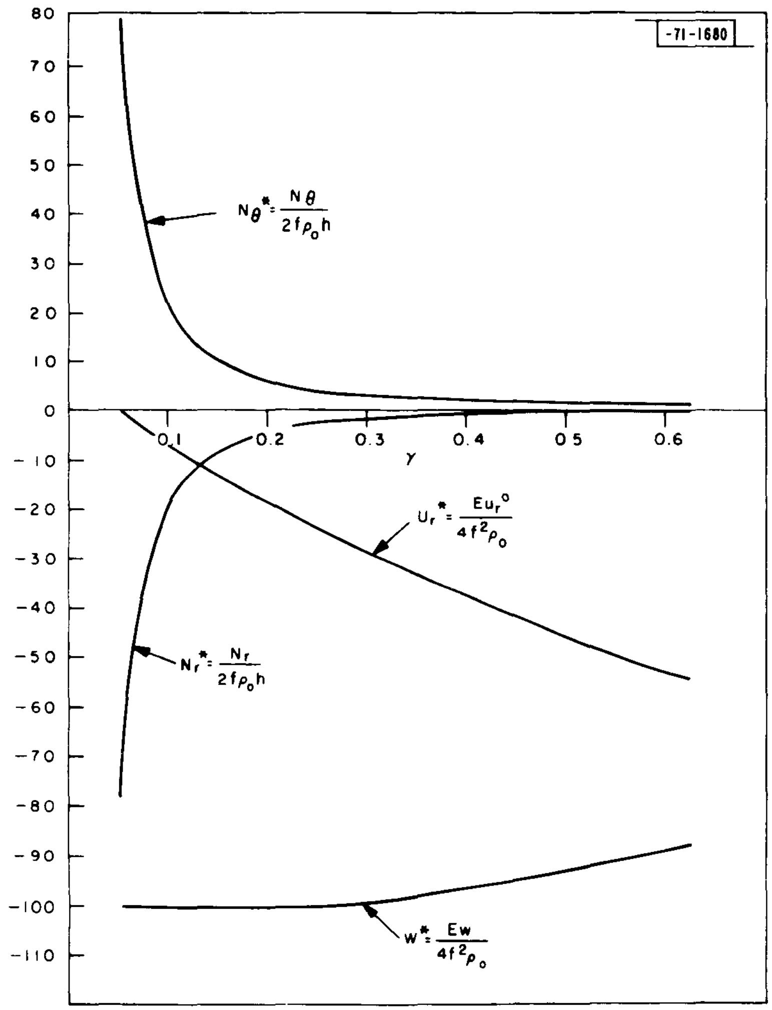


Case 6.4.2.14 Axi-Symmetric Gravity
Free γ_1 = 0.625
Restrained γ_2 = 0.0104

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0104	-1965.	1966.	.000	-2556.08
0.0208	-491.0	491.8	-39.	-2555
0.0624	-54.2	55 0	-.155	-2551.7
0.0937	-23.8	24.7	-235.6	-2545.7
0.1354	-11.23	12 03	-341.01	-2433.81
0.1458	-9.63	10.43	-367.00	-253.00
0.1874	-5.66	6.47	-469.67	-2513.23
0.2291	-3.65	4.46	-569.88	-2492.47
0.2708	-2.49	3.32	-667.33	-2468.22
0.3124	-1.75	2.60	-761.72	-2440.77
0.3541	-1.26	2.12	-852.78	-2410.50
0.3958	-.906	1.78	-940.32	-2377.74
0 4374	-.645	1.54	-1024.	-2342.86
0.4791	-.445	1.36	-1104.26	-2306.21
0 5208	-.286	1.22	-1180.54	-2268.12
0.5624	-.156	1.12	-1253.01	-2228.92
0 6041	-.048	1.04	-1321.70	-2188.88
0.625	.000	1.00	-1354.66	-2168.64

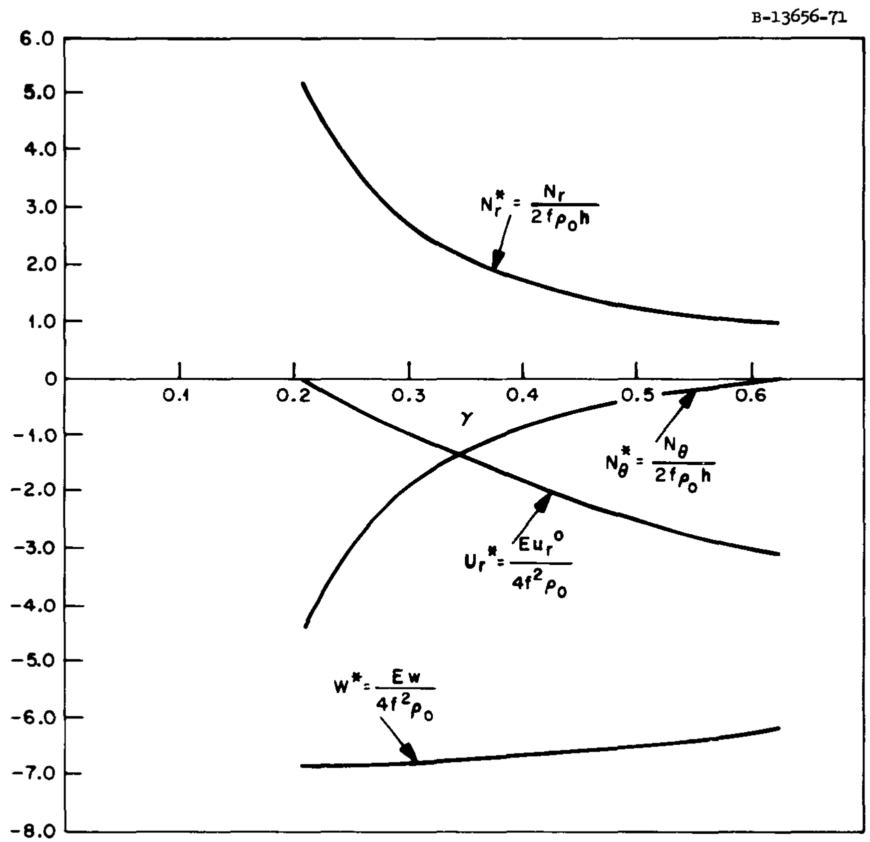


CASE 6.4.2.15 AXI-SYMMETRIC GRAVITY

Free $\gamma_1 = 0.625$

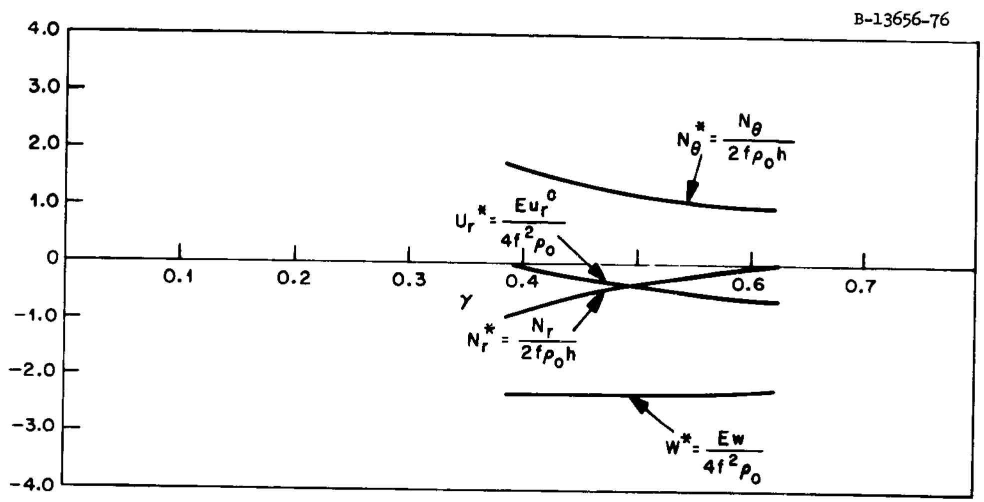
Restrained $\gamma_2 = 0.0521$

γ	N_r^*	N_θ^*	u_r^*	w^*
0.0520	-78.2	79.0	0.00	102.64
0.1041	-19.2	20.0	7.96	102.4
0.1458	- 9.62	10.4	12.9	101.99
0.1874	- 5.65	6.46	17.45	101.37
0.2291	- 3.64	4.46	21.74	100.59
0.2708	- 2.48	3.31	25.85	99.64
0.3124	- 1.75	2.59	29.78	98.58
0.3541	- 1.25	2.11	33.56	97.39
0.3958	- 0.905	1.78	37.16	96.10
0.4374	- 0.645	1.54	40.61	94.71
0.4791	- 0.444	1.36	43.89	93.26
0.5208	- 0.286	1.22	47.01	91.73
0.5624	- 0.156	1.11	49.96	90.16
0.625	0.000	1.000	54.10	87.75



Case 6.4.2.16 Axi-Symmetric Gravity
Free γ_1 = 0.6250
Restrained γ_2 = 0.2083

γ	N_r^*	N_θ^*	U_r^*	W^*
0.2083	-4.50	5.31	.000	-6.81
0.2187	-4.04	4.86	-.130	-6.81
0.2604	-2.72	3.55	-.594	-.679
0.3020	-1.90	2.74	-.995	-6.76
0.3437	-1.36	2.22	-1.352	-6.72
0.3854	-.983	1.85	-1.68	-6.66
0.4270	-.703	1.59	-1.97	-6.59
0.4687	-.490	1.40	-2.25	-6.51
0.5104	-.322	1.25	-2.50	-6.42
0.5520	-.186	1.14	-2.74	-6.33
0.5937	-.0735	1.05	-2.96	-6.24
0.625	.000	.999	-3.11	-6.16



Case 6.4.2.17 Axi-Symmetric Gravity
Free γ_1 = 0.6250
Restrained γ_2 = 0.3854

γ	N_r^*	N_θ^*	U_r^*	w^*
0.3854	-.983	1.85	.000	-2.30
0.3958	-.905	1.78	-.037	-2.30
0.4374	-.645	1.54	-.176	-2.30
0.4791	-.444	1.36	-.300	-2.28
0.5208	-.286	1.22	-.411	-2.27
0.5624	-.156	1.11	-.510	-2.24
0.6041	-.0480	1.03	-.600	-2.22
0.625	.000	.999	-.642	-2.21

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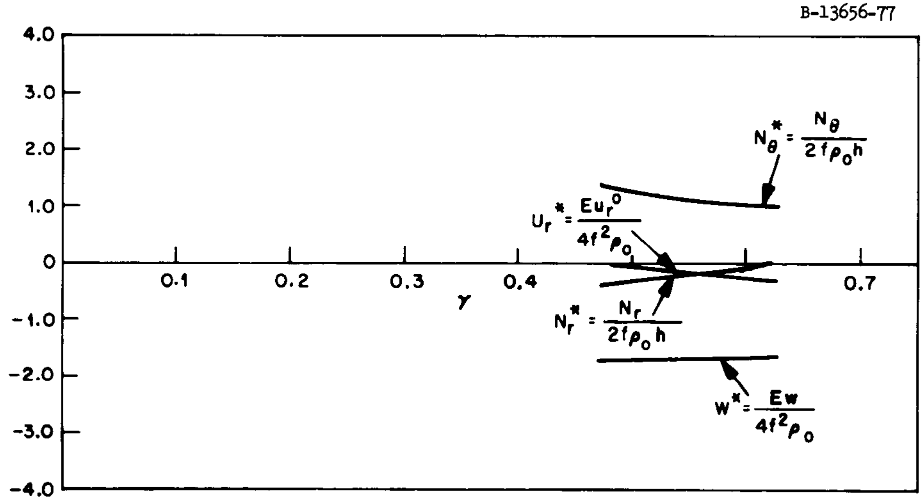
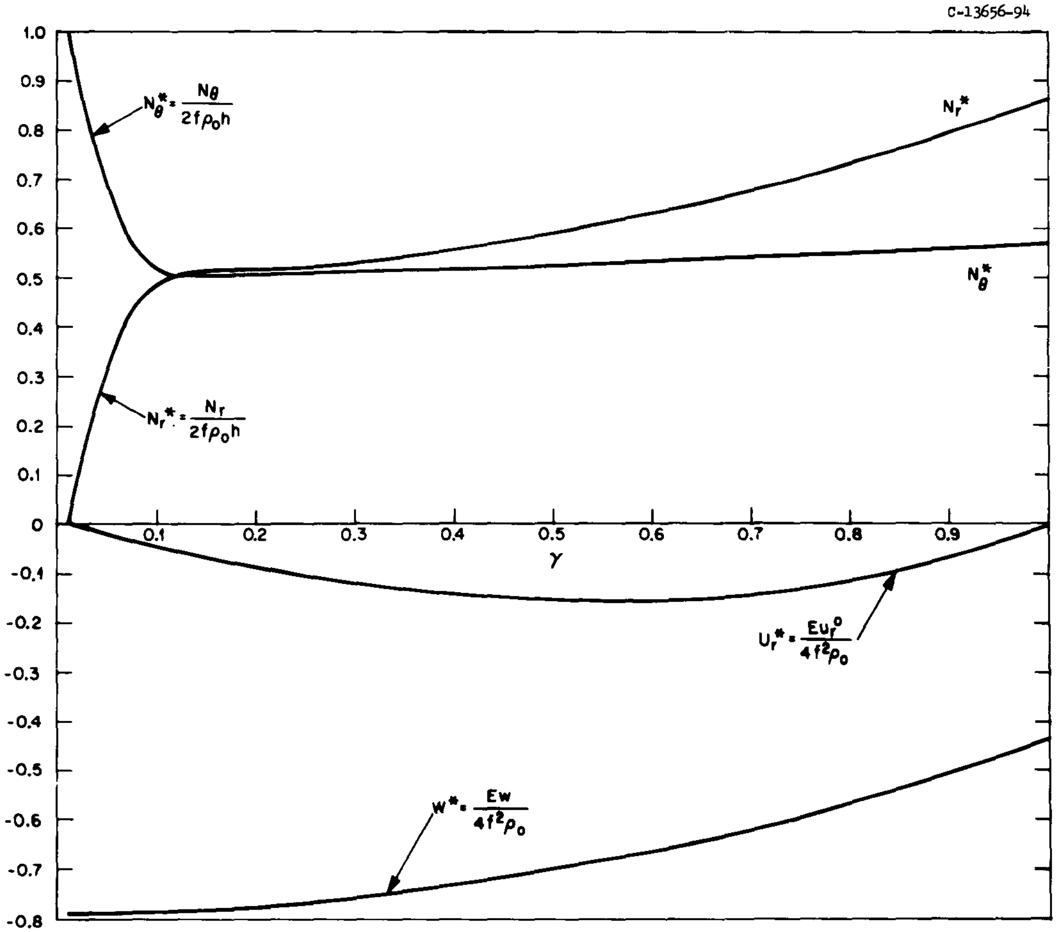


Case 6.4.2.18 Axi-Symmetric Gravity
Free γ_1 = 0.6250
Restrained γ_2 = 0.4687

γ	N_r^*	N_θ^*	U_r^*	w^*
0.5687	-.490	1.40	.000	-1.71
0.4791	-.444	1.36	-.025	-1.71
0.5208	-.286	1.22	-.116	-1.70
0.5624	-.156	1.11	-.198	-1.69
0.6041	-.0480	1.03	-.271	-1.67
0.625	.000	.999	-.304	-1.67



Case 6.4.2.19 Axi-Symmetric Gravity
Free γ_1 = 0.0104
Restrained γ_2 = 0.999

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0104	0.	0.999	0.0021	-0.7893
0.1145	.500	.505	-0.0489	-0.7847
0.2187	0.5167	.506	-0.0901	-0.7721
0.3229	-.5383	0.5124	-0.1237	-0.7519
0.4270	0.567	0.520	-0.1469	-0.7244
0.4312	0.604	0.528	-0.1574	-0.6898
0.6354	0.648	-.538	-0.1533	-0.6481
0.7395	0.700	0.547	-0.1328	-0.5990
0.8437	0.759	0.556	-0.0945	-0.5418
0.9479	0.8259	0.5649	-0.0367	-0.4758
0.999	0.8618	0.5690	-0.000	-0.4391

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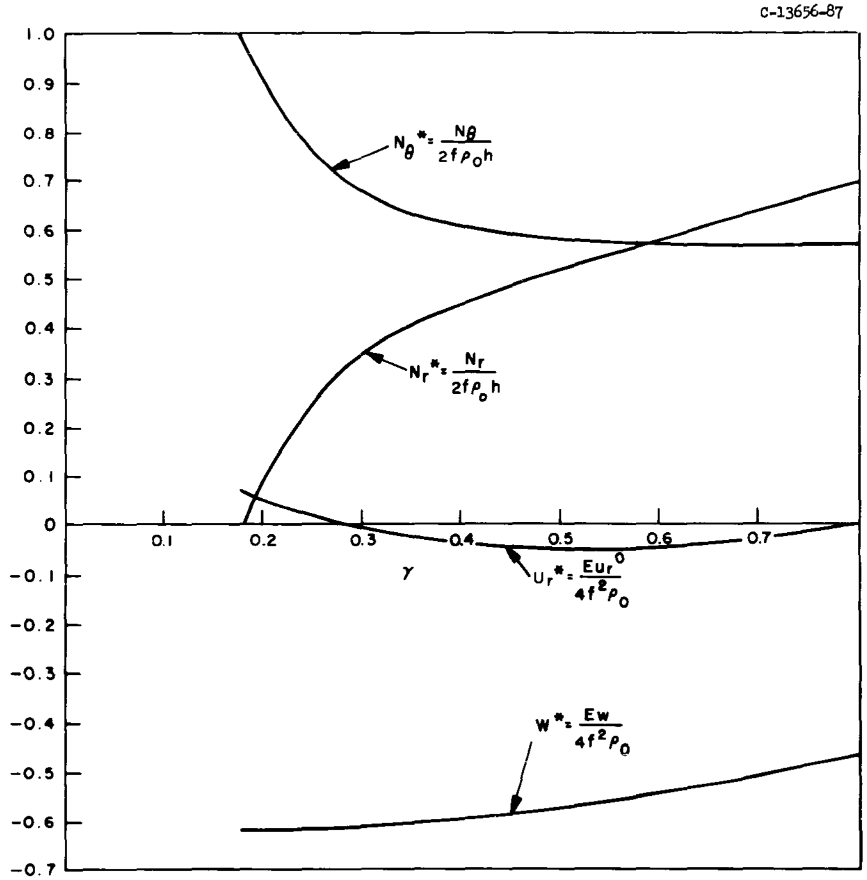
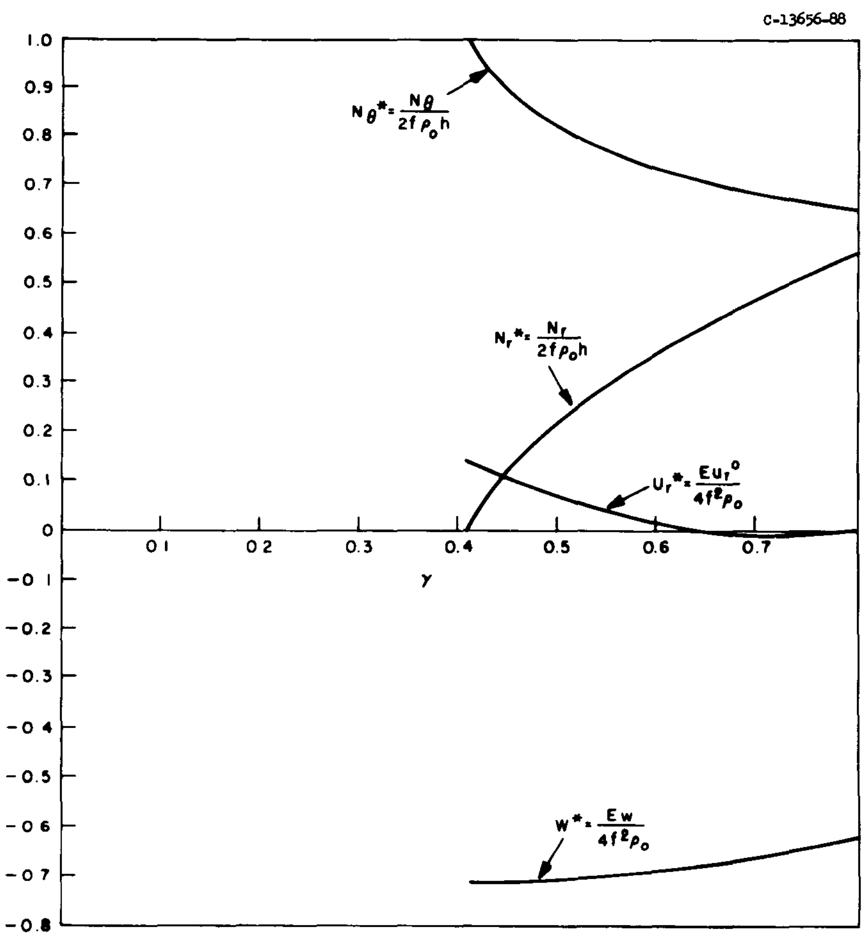


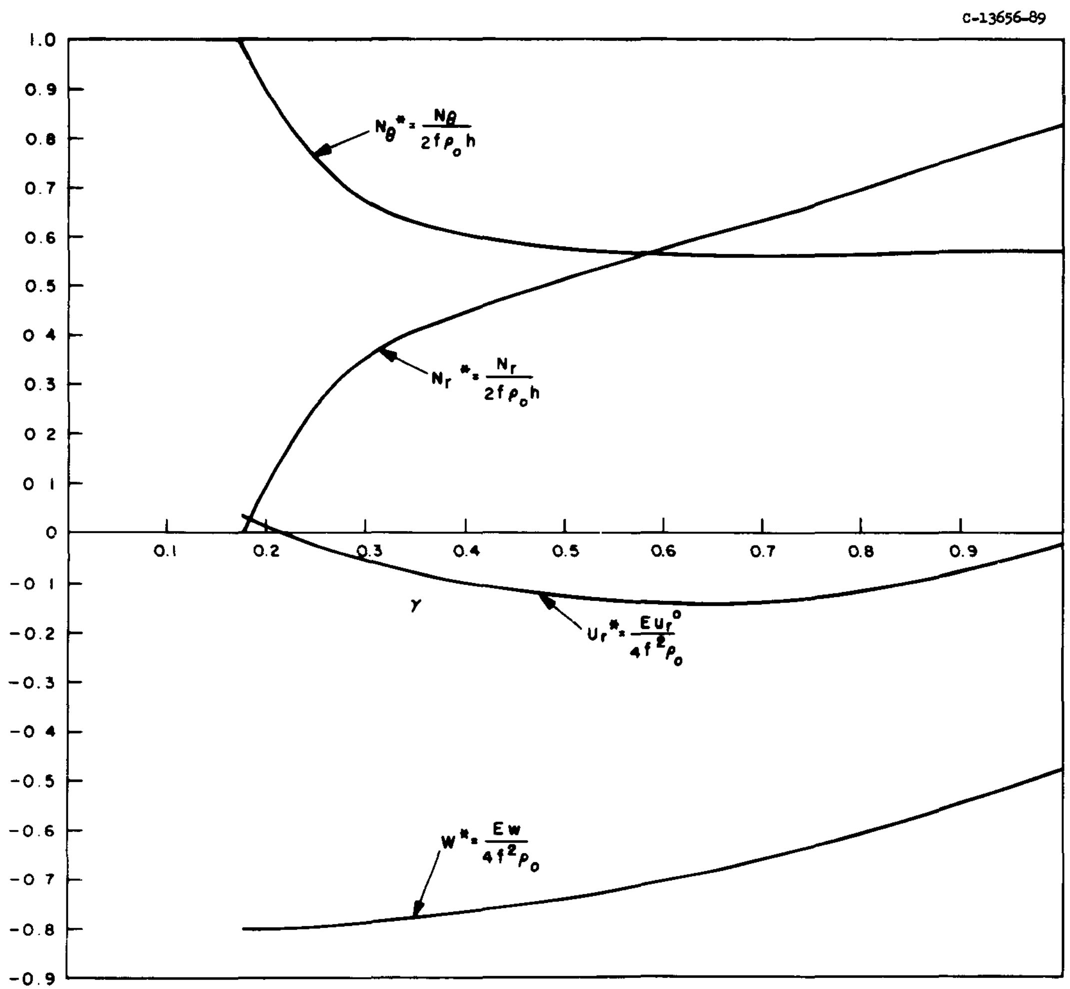
Case 6.4.2.20 Axi-Symmetric Gravity
Free γ_1 = 0.1785
Restrained γ_2 = 0.8035

γ	N_r^*	N_θ^*	U_r^*	W^*
0.1785	0.	.999	.0714	-.6156
0.2678	.294	.724	.0123	-.6126
0.3571	.413	.633	-.0221	-.6024
0.4464	.485	.595	-.0420	-.5862
0.5357	.542	.578	-.0497	-.5642
0.6249	.595	.571	-.0455	-.5365
0.7142	.648	.570	-.0291	-.5029
0.8035	.703	.572	-.000	-.4631



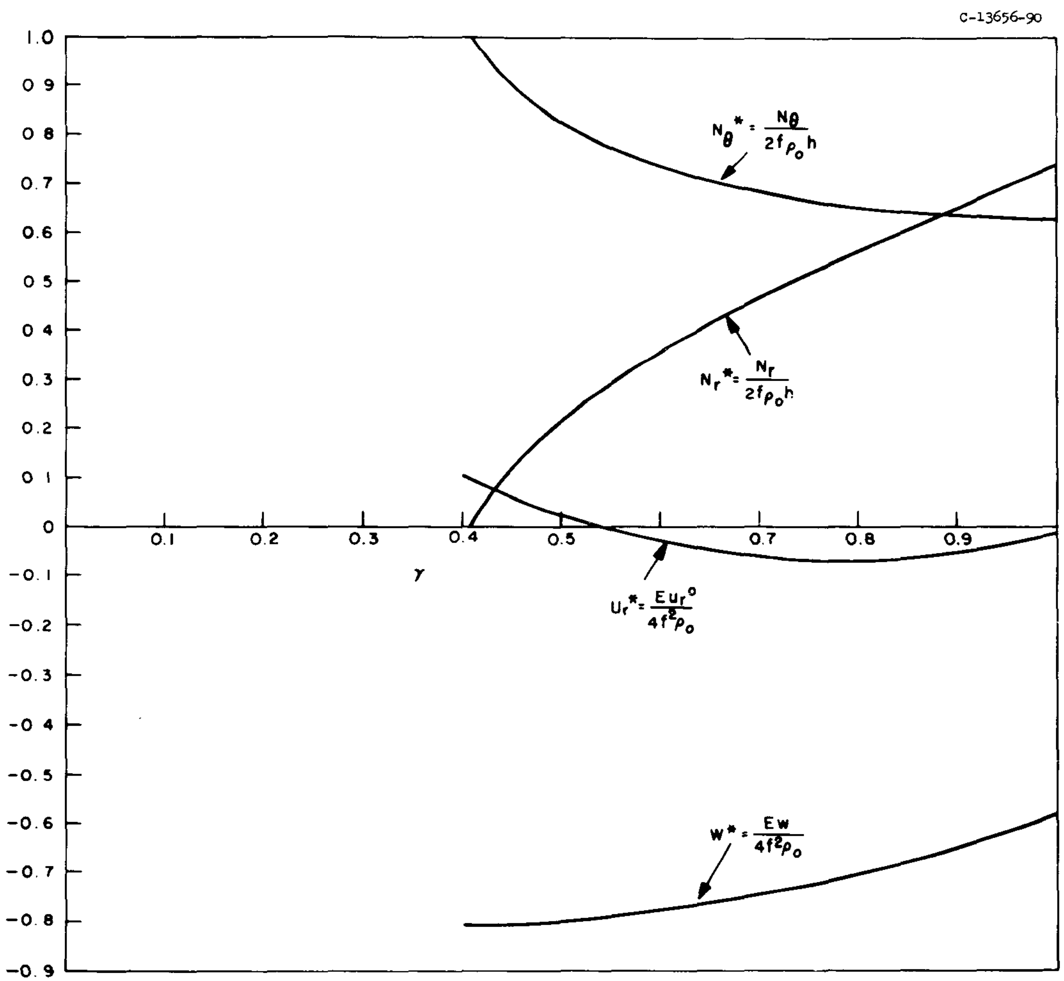
Case 6.4.2.21 Axi-Symmetric Gravity
Free γ_1 = 0.4017
Restrained γ_2 = 0.8035

γ	N_r^*	N_θ^*	U_r^*	W^*
0.4017	0.	.999	.145	-.716
0.4464	.113	.905	.107	-.714
0.4910	.201	.837	.0761	-.710
0.5357	.274	.786	.0509	-.703
0.5803	.336	.748	.0310	-.695
0.6249	.390	.719	.0160	-.684
0.6696	.439	.696	.0055	-.671
0.7142	.485	.678	-.0006	-.656
0.7589	.528	.664	-.0024	-.639
0.8035	.569	.654	.000	-.620



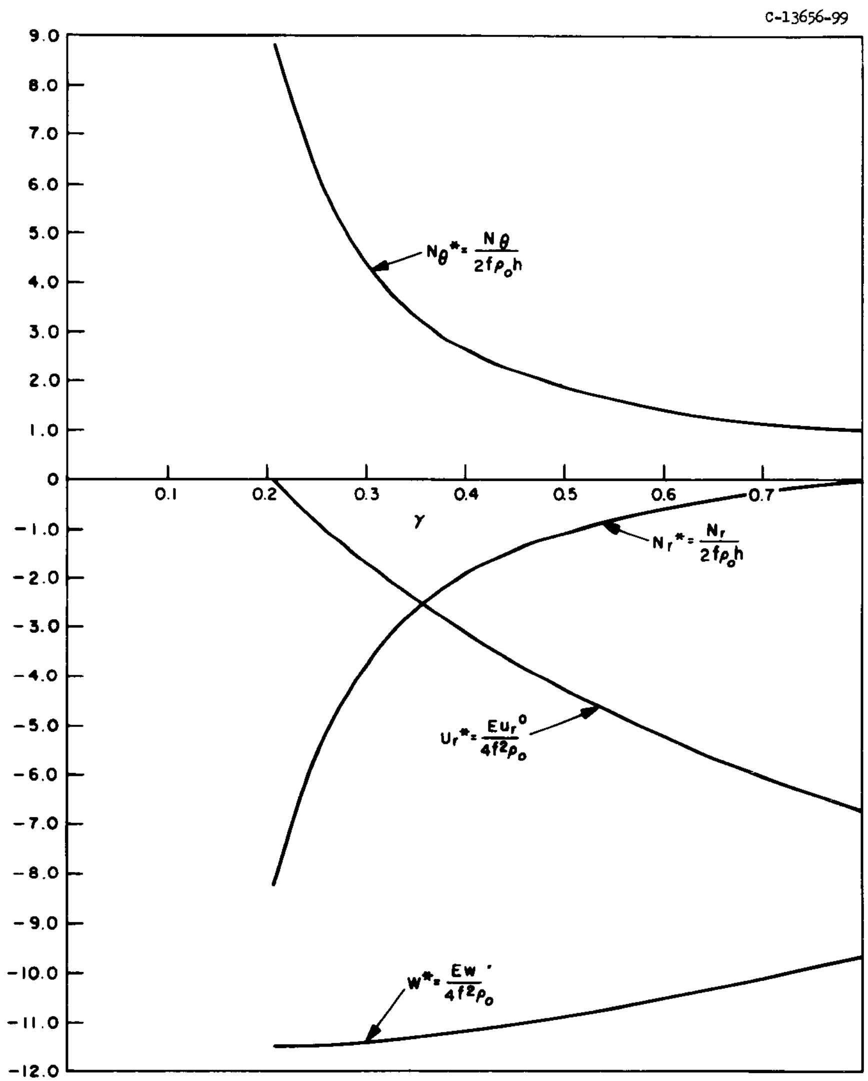
Case 6.4.2.22 Axi-Symmetric Gravity
Free $\gamma_1 = 0.1785$
Restrained $\gamma_2 = 1.0267$

γ	N_r^*	N_θ^*	U_r^*	W^*
0.1785	0.	.999	.0383	-.800
0.2678	.294	.724	-.0363	-.794
0.3571	.413	.633	-.0854	-.779
0.4464	.485	.595	-.118	-.758
0.5357	.542	.578	-.138	-.730
0.6249	.595	.571	-.145	-.696
0.7142	.648	.570	-.138	-.656
0.8035	.703	.572	-.117	-.609
0.8928	.762	.575	-.0825	-.557
0.9821	.826	.579	-.0316	-.497
1.0267	.859	.581	-.000	-.464



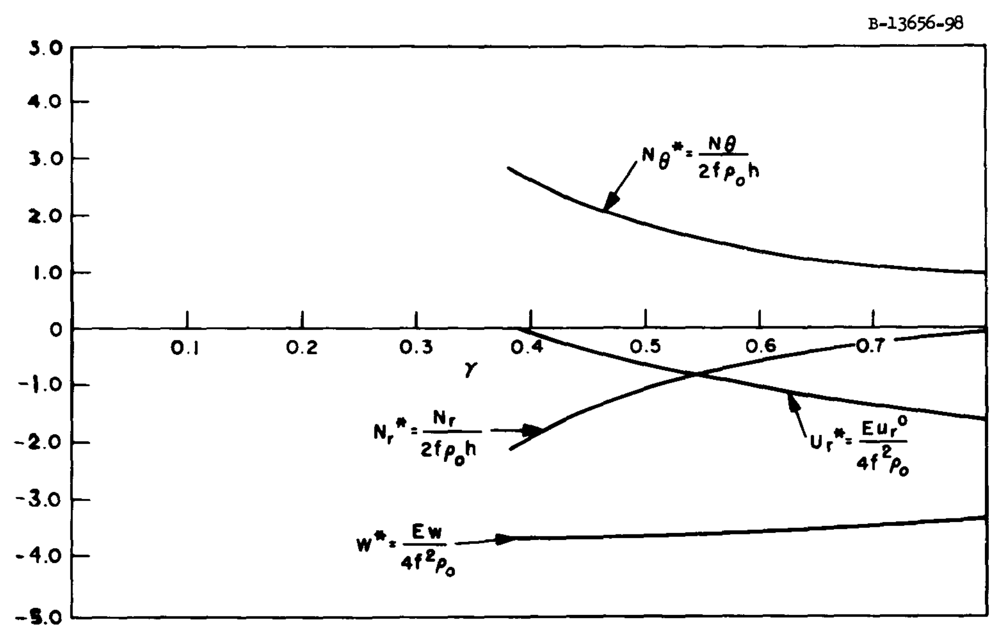
Case 6.4.2.23 Axi-Symmetric Gravity
Free $\gamma_1 = 0.4017$
Restrained $\gamma_2 = 1.0267$

γ	N_r^*	N_θ^*	U_r^*	w^*
0.4017	0.	.999	.105	-.815
0.4910	.201	.837	.0291	-.806
0.5803	.336	.748	-.0224	-.787
0.6696	.439	.696	-.0538	-.7598
0.7589	.528	.664	-.0669	-.724
0.8482	.609	.645	-.0623	-.680
0.9374	.688	.633	-.0401	-.628
1.0267	.767	.626	-.000	-.568



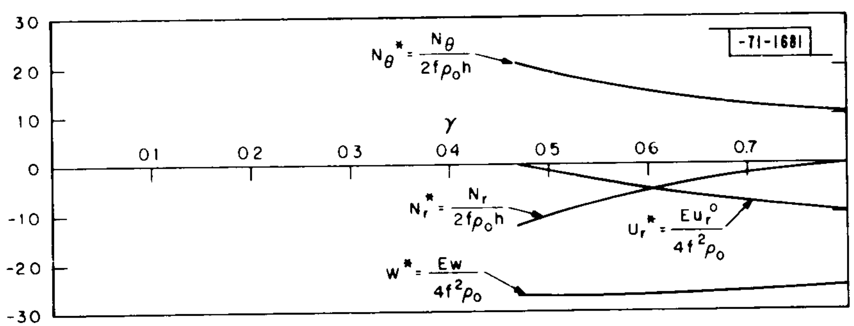
Case 6.4.2.24 Axi-Symmetric Gravity
Free γ_1 = 0.8020
Restrained γ_2 = 0.2083

γ	N_r^*	N_θ^*	U_r^*	W^*
0.2083	-8.165	8.825	.000	-11.517
0.2499	-5.560	6.233	-.838	-11.500
0.2916	-3.985	4.672	-1.557	-11.453
0.3333	-2.9580	3.662	-2.195	-11.381
0.3749	-2.249	2.971	-2.772	-11.289
0.4166	-1.737	2.480	-3.302	-11.178
0.4583	-1.353	2.118	-3.792	-11.052
0.4999	-1.057	1.845	-4.247	-10.913
0.5416	-0.821	1.635	-4.671	-10.763
0.5833	-0.629	1.469	-5.066	-10.605
0.6249	-0.469	1.337	-5.434	-10.439
0.6666	-0.334	1.231	-5.777	-10.267
0.7083	-0.216	1.144	-6.097	-10.090
0.7499	-0.113	1.072	-6.395	-9.910
0.8020	-.000	1.000	-6.738	-9.682



Case 6.4.2.25 Axi-Symmetric Gravity
Free γ_1 = 0.8020
Restrained γ_2 = 0.3854

γ	N_r^*	N_θ^*	U_r^*	W^*
0.3854	-2.106	2.833	.000	-3.713
0.4270	-1.631	2.379	-0.251	-3.708
0.4687	-1.272	2.043	-.475	-3.696
0.5104	-.993	1.787	-.678	-3.671
0.5520	-.769	1.589	-.862	-3.642
0.5937	-.586	1.433	-1.030	-3.607
0.6354	-.433	1.309	-1.183	-3.568
0.6770	-.303	1.207	-1.324	-3.524
0.7187	-.189	1.125	-1.452	-3.476
0.7605	-.0897	1.056	-1.569	-3.425
0.8020	-.000	1.000	-1.676	-3.371



CASE 6.4.2.26 AXI-SYMMETRIC GRAVITY

Free $\gamma_1 = 0.8021$

Restrained $\gamma_2 = 0.4688$

γ	N_r^*	N_θ^*	u_r^*	w^*
0.4687	-1.272	2.0433	0.0	-2.678
0.4895	-1.124	1.907	-0.0866	-2.676
0.5104	-0.993	1.787	-0.168	-2.672
0.5312	-0.875	1.682	-0.246	-2.667
0.5520	-0.769	1.589	-0.320	-2.660
0.5729	-0.673	1.507	-0.391	-2.652
0.5937	-0.586	1.433	-0.458	-2.643
0.6145	-0.507	1.368	-0.521	-2.633
0.6354	-0.433	1.309	-0.582	-2.622
0.6562	-0.366	1.255	-0.640	-2.609
0.6770	-0.303	1.207	-0.695	-2.596
0.6979	-0.244	1.164	-0.748	-2.581
0.7187	-0.189	1.125	-0.798	-2.566
0.7395	-0.138	1.0894	-0.845	-2.550
0.7604	-0.0897	1.0568	-0.891	-2.533
0.7812	-0.0437	1.0271	-0.934	-2.515
0.8020	-0.000	1.000	-0.974	-2.497

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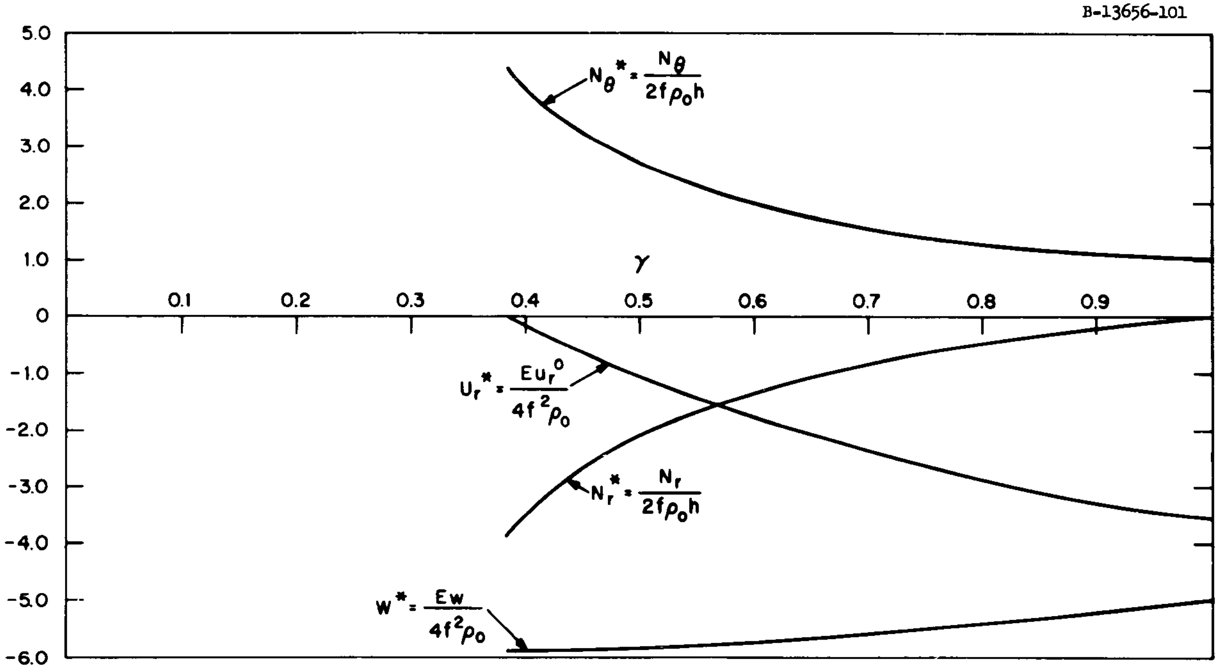
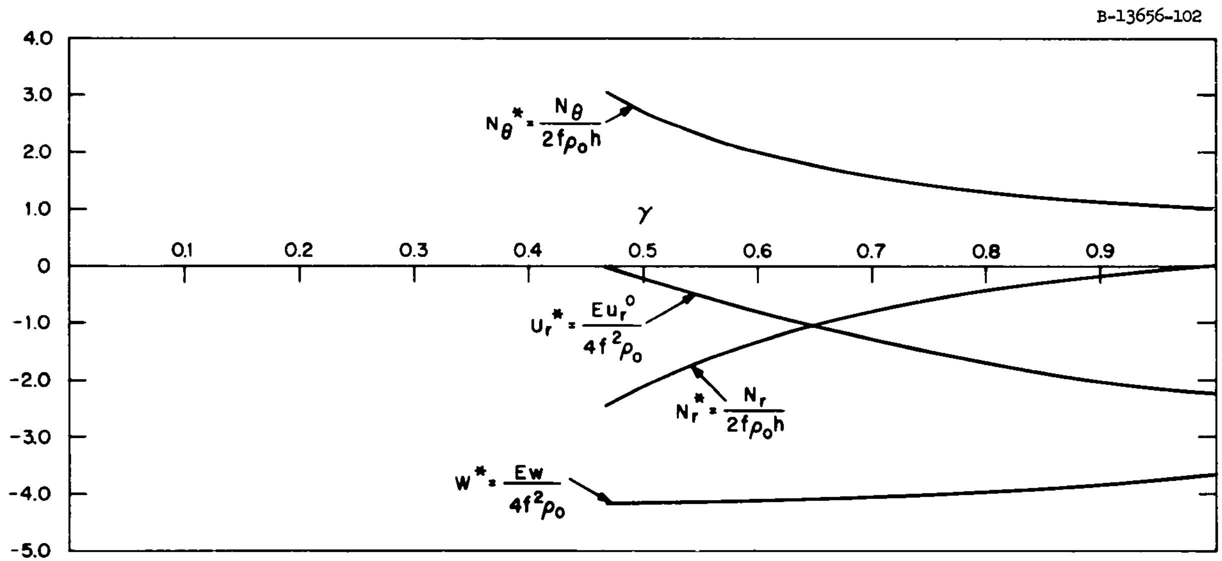


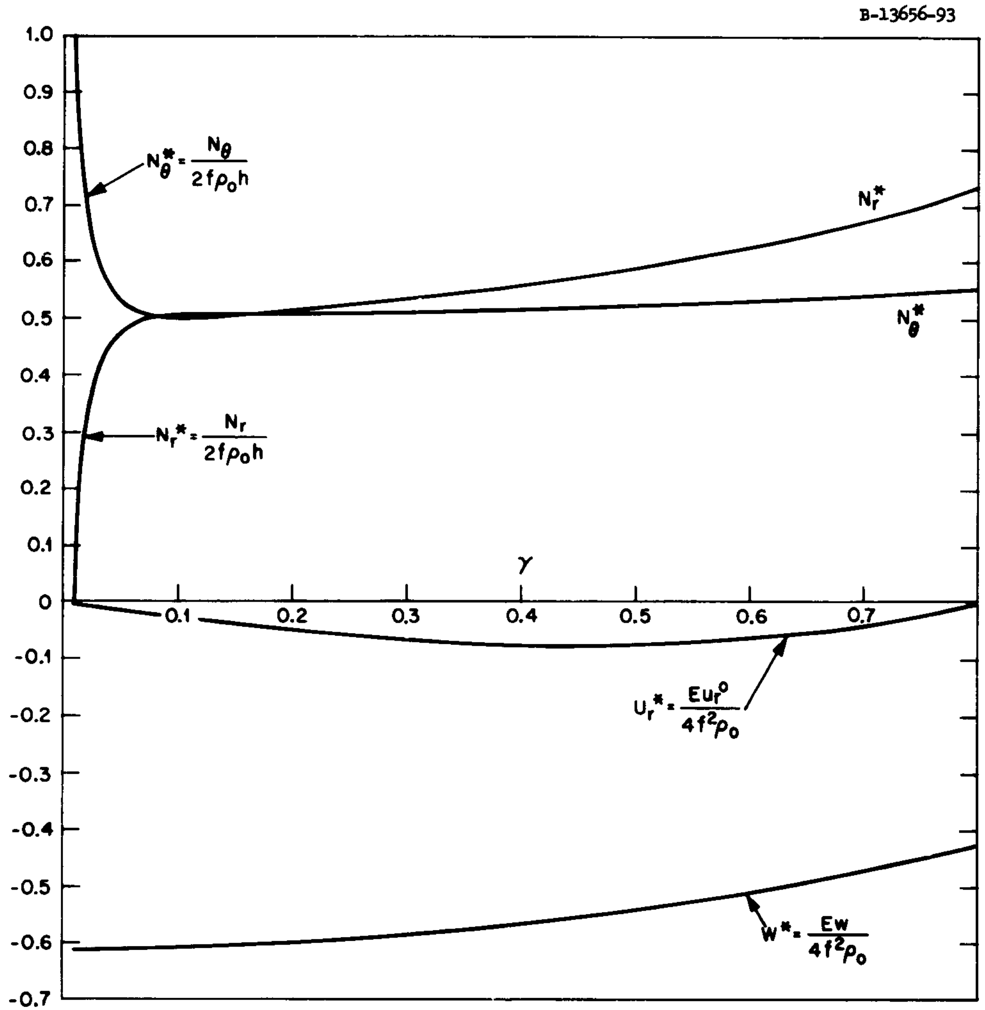
Case 6.4.2.27 Axi-Symmetric Gravity
Free $\gamma_1 = 0.999$
Restrained $\gamma_2 = 0.3854$

γ	N_r^*	N_θ^*	U_r^*	W^*
0.3854	-3.841	4.344	.000	-5.891
0.4270	-3.065	3.592	-.418	-5.886
0.4687	-2.481	3.034	-.793	-5.866
0.5104	-2.030	2.610	-1.133	-5.835
0.5520	-1.671	2.281	-1.444	-5.793
0.5937	-1.380	2.020	-1.728	-5.743
0.6354	-1.139	1.811	-1.991	-5.685
0.6770	-.937	1.642	-2.233	-5.621
0.7187	-.763	1.503	-2.456	-5.551
0.7604	-.612	1.388	-2.663	-5.477
0.8020	-.479	1.291	-2.854	-5.398
0.8437	-.3605	1.210	-3.030	-5.316
0.8854	-.2531	1.141	-3.192	-5.231
0.9270	-.154	1.083	-3.341	-5.144
0.9687	-.0640	1.033	-3.477	-5.054
0.999	-.000	1.000	-3.572	-4.986



Case 6.4.2.28 Axi-Symmetric Gravity
Free $\gamma_1 = 1.000$
Restrained $\gamma_2 = 0.4687$

γ	N_r^*	N_θ^*	U_r^*	W^*
0.4687	-2.481	3.034	.000	-4.173
0.5312	-1.840	2.435	-.415	-4.164
0.5937	-1.380	2.020	-.774	-4.136
0.6562	-1.034	1.722	-1.089	-4.091
0.7187	-.763	1.503	-1.365	-4 033
0.7812	-.543	1.337	-1.609	-3.965
0.8437	-.360	1.210	-1.824	-3.887
0.9062	-.202	1.111	-2.012	-3.803
0.9687	-.064	1.033	-2.176	-3.711
1.000	0.	.999	-2.250	-3.664

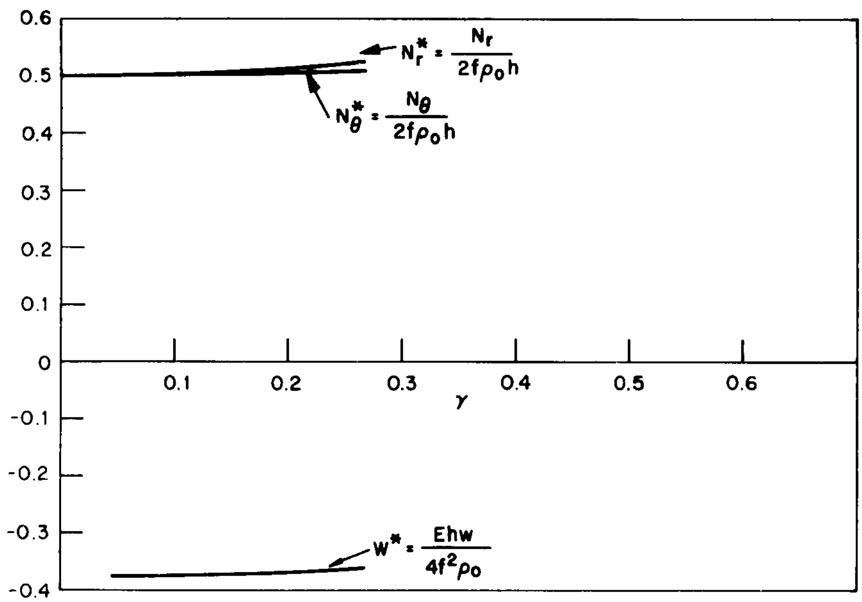


Case 6.4.2.29 Axi-Symmetric Gravity
Free γ_1 = 0.0104
Restrained γ_2 = 0.8020

γ	N_r^*	N_θ^*	u_r^*	w^*
0.0104	0.	0.999	0.0040	-0.6101
0.0937	.497	.507	-.0232	-.6079
0.1770	.509	.505	-.0431	-.6016
0.2604	.524	.508	-0.594	-.5915
0.3437	.543	.513	-.0710	-.577
0.4270	.567	.520	-0.0765	-.5596
0.5104	.596	.526	-.0749	-.5377
0.5937	.629	.534	-.0653	-.5116
0.6770	.668	.541	-.0467	-.4810
0.7604	.711	.549	-0.0182	-.4456
0.8020	.734	.552	-.000	-.4259

6.4.3 SHELL CLOSED AT THE APEX

The case in which the shell is closed at the apex is best handled by considering the equilibrium of a portion of the shell bounded by the apex and a circle of latitude (γ =constant). This is illustrated in figure 6.4.3.1.



Case 6.4.3.1 Axi-Symmetric Gravity
Closed at Apex
Restrained $\gamma_2 = 0.2678$

γ	N_r^*	N_θ^*	U_r^*	W^*
0	.500	.500	0	---
0.0446	.500	.500	-0.0010	-0.374
0.0669	.501	.500	-0.0015	-0.374
0.0892	.502	.500	-0.0019	-0.373
0.1116	.504	.501	-0.0022	-0.372
0.1339	.506	.502	-0.002	-0.371
0.1562	.509	.502	-0.0025	-0.370
0.1785	.511	.503	-0.0024	-0.369
0.2008	.515	.504	-0.0021	-0.368
0.2232	.518	.505	-0.0016	-0.366
0.2455	0.522	.507	-0.0009	-0.364
0.2678	.526	.508	0.000	-0.363

Figure 6.4.3.1

The force equilibrium equation of the portion of the shell shown in figure 6.4.3.1 is written as

$$N_r \sin\beta \int_0^{2\pi} 2f\,\gamma\, d\theta = \int_0^{2\pi}\int_0^{\gamma} 2f\rho_o h \sqrt{1+(\gamma')^2}\, d\gamma' \cdot 2f\gamma' d\theta. \qquad 6.4.3.1$$

This results in

$$N_r(r) = \frac{2f\rho_0 h}{3} \quad \frac{\sqrt{1+(r)^2}}{(r)^2} \left\{ \left[1+(r)^2 \right]^{3/2} - 1 \right\}. \qquad 6.4.3.2$$

At the apex, $r = 0$, equation 6.4.3.2 yields an indeterminate value for $N_r(o)$ since both the numerator and denominator become zero. The application of L'Hospital's Rule to equation 6.4.3.2 yields

$$N_r(o) = f\rho_0 h. \qquad 6.4.3.3$$

If equation 6.4.3.2 is compared to equation 6.4.14 then it is clear that

$$C_1 = -1. \qquad 6.4.3.4$$

If the upper boundary (r_2) of the shell is restrained in the tangential direction such that

$$u_r^o(r_2) = 0, \qquad 6.4.3.5$$

then as in case 6.4.2

$$C_2 = \left\{ (1+\nu)\left[\frac{\sqrt{1+(\gamma_2)^2}}{(\gamma_2)^2} + \ln\left(\frac{1+\sqrt{1+(\gamma_2)^2}}{\gamma_2}\right)\right] - \sqrt{1+(\gamma_2)^2} \right\} C_1$$
$$- (1+\nu)\ln\gamma_2 + \frac{(1+\nu)}{(\gamma_2)^2} - \frac{(3-\nu)}{2}(\gamma_2)^2 - \frac{1}{4}(\gamma_2)^4. \qquad 6.4.3.6$$

Note that $u_r^0(\gamma)$ will also be finite at $\gamma = 0$ by L'Hospital's rule.

Six cases have been solved for the shell which is closed at the apex. The value of w* at the apex ($\gamma = 0$) cannot be calculated directly from the solution and hence has not been included. However, it is very easy to extrapolate values of w* near the apex to obtain the value at the apex.

The following table lists the outer boundary of the cases which have been considered.

Table 6.4.3.1

Case	Restrained at γ_2
6.4.3.1	.2678
6.4.3.2	.3854
6.4.3.3	.5357
6.4.3.4	.6250
6.4.3.5	.8035
6.4.3.6	1.0267

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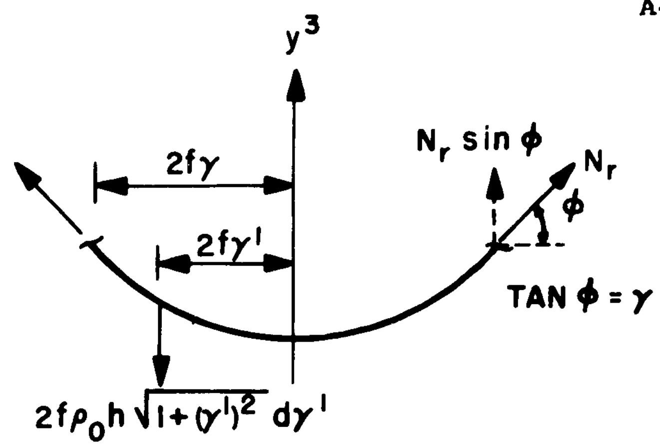

y^3
$2f\gamma$
$2f\gamma'$
$N_r \sin\phi$
N_r
ϕ
TAN $\phi = \gamma$
$2f\rho_0 h\sqrt{1+(\gamma')^2}\,d\gamma'$

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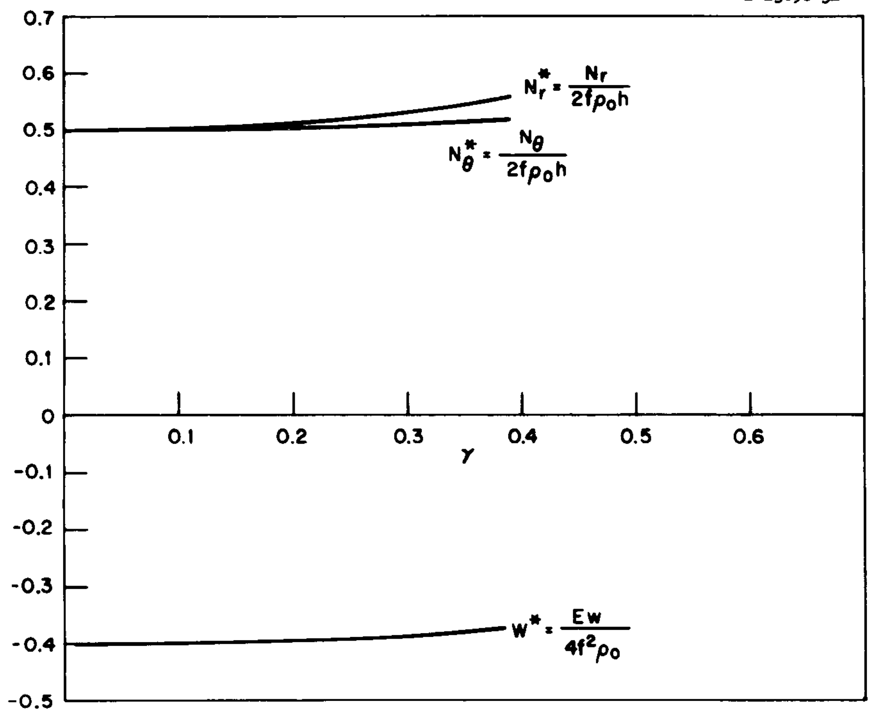
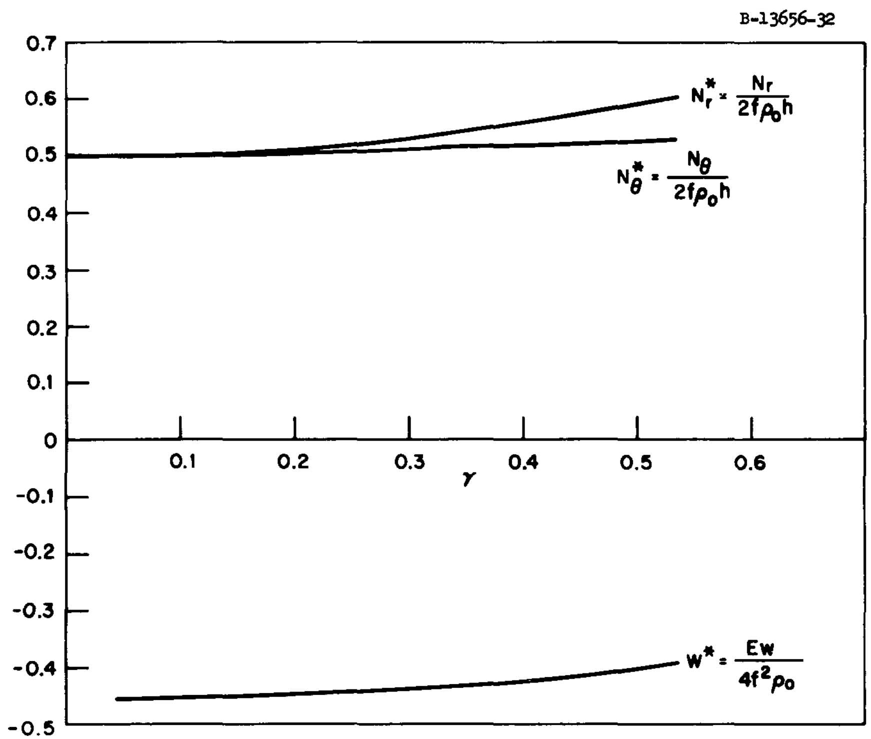


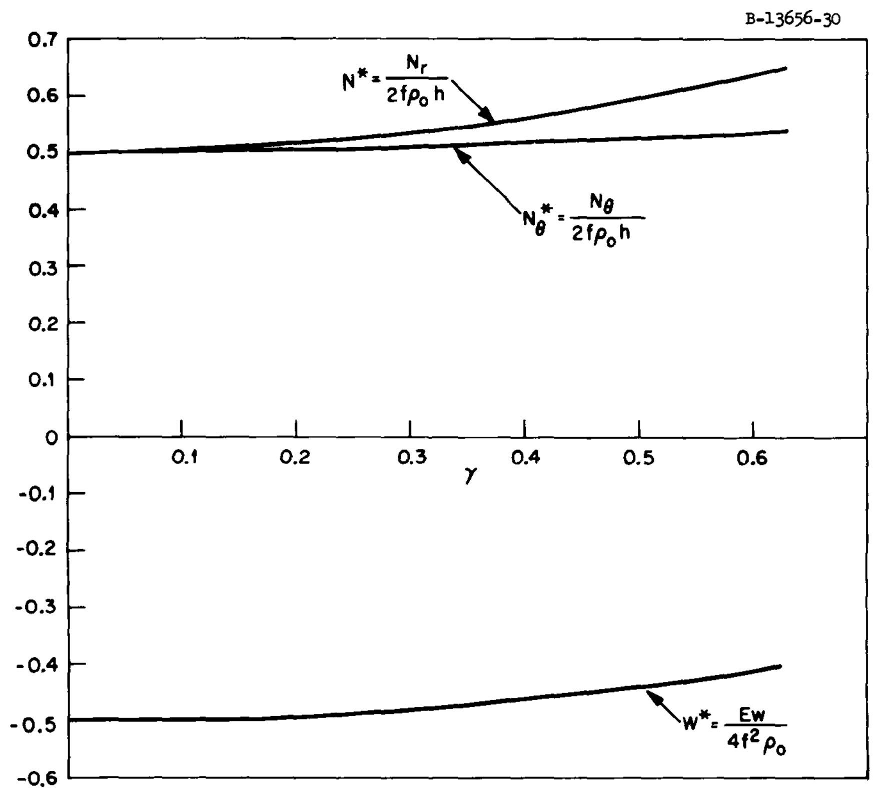
Case 6.4.3.2 Axi-Symmetric Gravity
Closed at Apex
Restrained $\gamma_2 = 0.3854$

γ	N_r^*	N_θ^*	u_r^*	w^*
0	.500	.500	0	---
0.0520	.501	.500	-.00268	-.402
0.0937	.503	.501	-.0046	-.401
0.1354	.507	.502	-.0062	-.399
0.1170	.512	.504	-.0073	-.397
0.2187	.518	.506	-.0077	-.394
0.2604	.525	.508	-.0073	-.390
0.3020	.534	.510	-.0060	-.386
0.3437	.544	.513	-.0036	-.381
0.3854	.555	.516	.000	-.375



Case 6.4.3.3 Axi-Symmetric Gravity
Closed at Apex
Restrained $\gamma_2 = 0.5357$

γ	N_r^*	N_θ^*	U_r^*	w^*
0	0.500	0.500	0	---
0.6696	0.501	0.500	-0.006	-0.454
0.1116	0.504	0.501	-0.011	-0.453
0.1562	0.509	0.502	-0.015	-0.450
0.2008	0.515	0.504	-0.018	-0.447
0.2455	0.5225	0.507	-0.020	-0.443
0.2901	0.5314	0.509	-0.021	-0.438
0.3348	0.541	0.512	-0.021	-0.432
0.3794	0.553	0.516	-0.019	-0.425
0.4241	0.566	0.519	-0.016	-0.417
0.4687	0.581	0.523	-0.011	-0.409
0.5133	0.597	0.527	-0.004	-0.399



Case 6.4.3.4 Axi-Symmetric Gravity
Closed at Apex
Restrained $\gamma_2 = 0.625$

γ	N_r^*	N_θ^*	U_r^*	w^*
0	.500	.500	0	---
0.1091	.501	.501	-.0156	-.495
0.1951	.514	.505	-.0259	-.490
0.2810	.529	.509	-.0327	-.480
0.3670	.550	.515	-.0347	-.467
0.4530	.576	.522	-.0306	-.451
0.5390	.607	.529	-.0194	-.430
0.6249	.644	.537	.000	-.405

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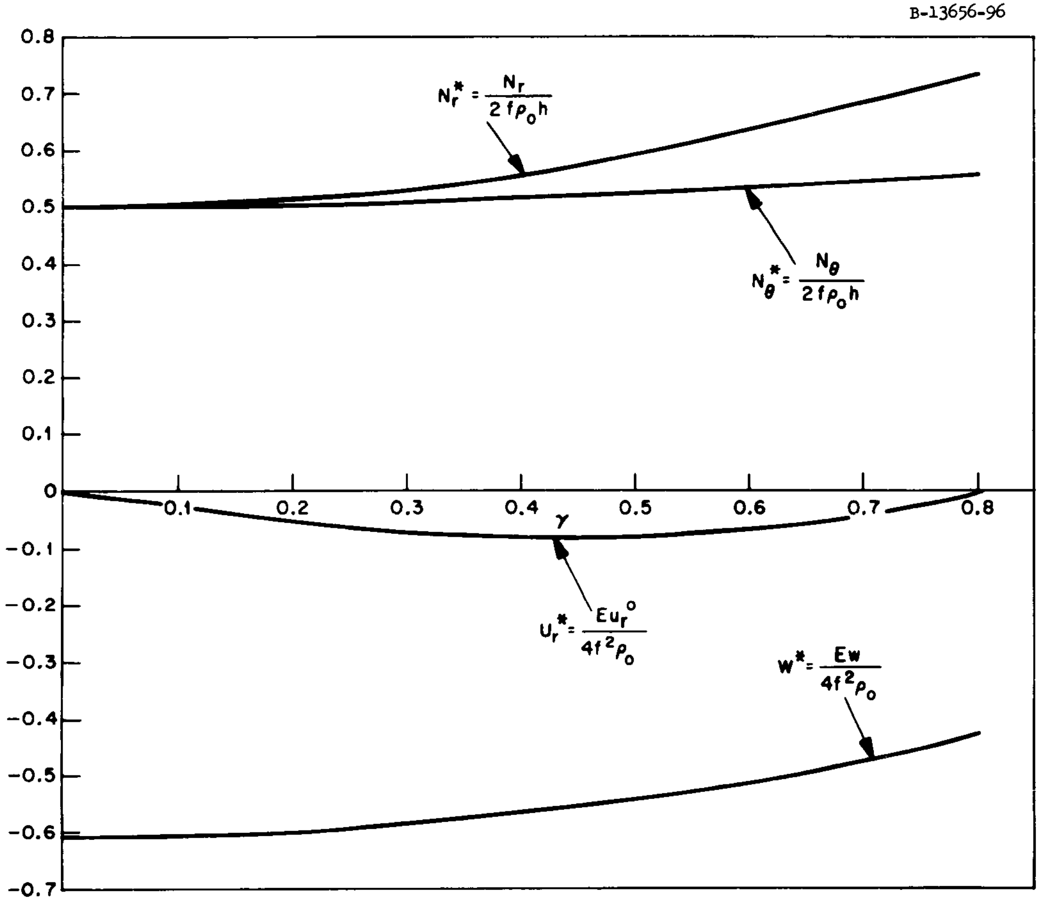
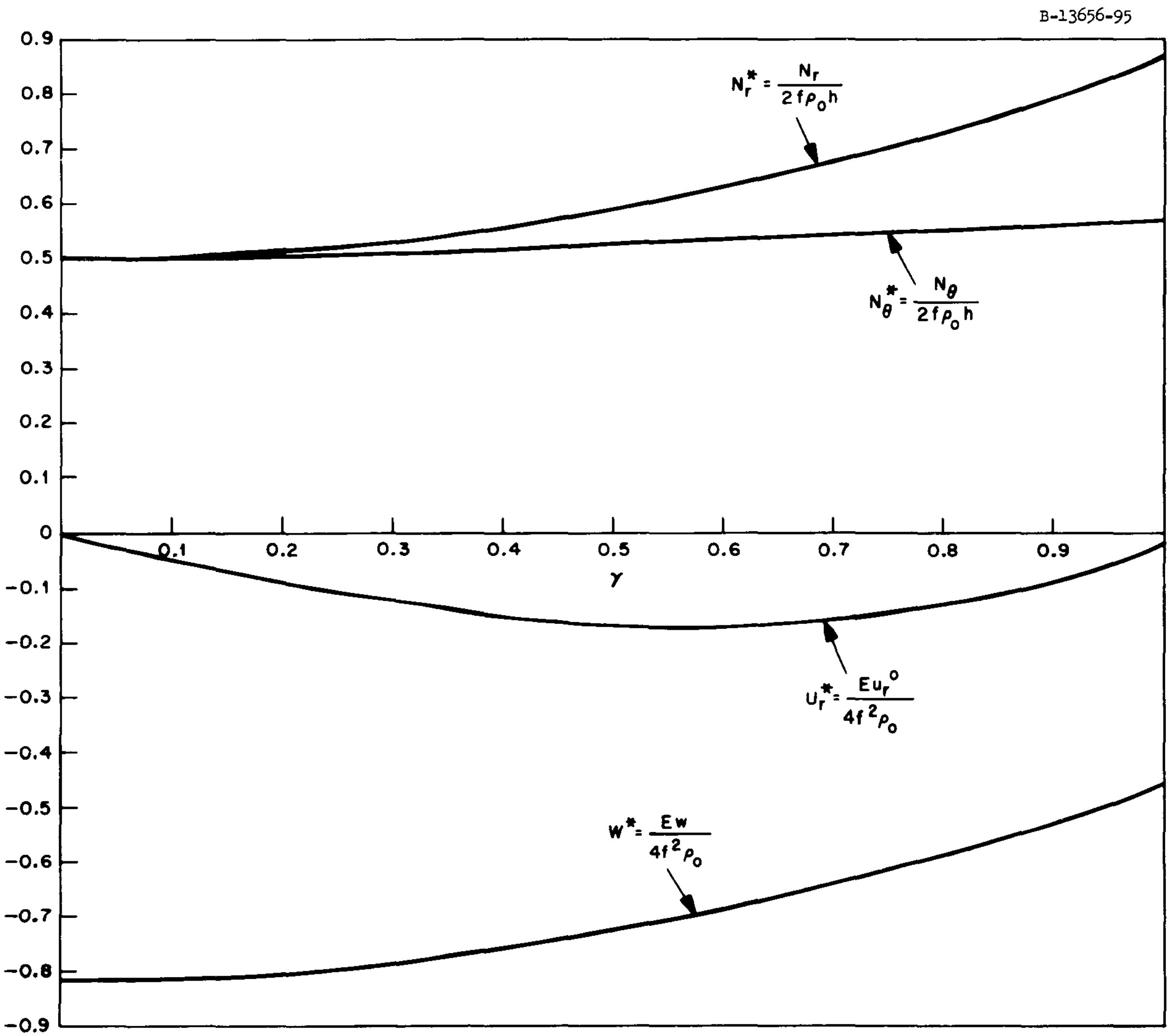


Case 6.4.3.5 Axi-Symmetric Gravity
Closed at Apex
Restrained γ_2 =0.8035

γ	N_r^*	N_θ^*	U_r^*	w^*
0	.500	.500	0	---
0.0892	.502	.500	-0.0230	-0.6092
0.1785	.511	.503	-0.0440	-0.6025
0.2678	.526	.508	-0.0612	-0.5914
0.3571	.547	.514	-0.0728	-0.5758
0.4464	.573	.521	-0.0774	-0.5558
0.5357	.606	.529	-0.0735	-0.5311
0.6249	.643	.536	-0.0600	-0.5015
0.7142	.687	.544	-0.0359	-0.4666
0.8035	.735	.552	-0.000	-0.4260



Case 6.4.3.6 Axi-Symmetric Gravity
Closed at Apex
Restrained $\gamma_2 = 1.0267$

γ	N_r^*	N_θ^*	U_r^*	w^*
0	.500	.500	0	---
0.0446	.500	.500	-0.0208	-0.8182
0.1339	.506	.502	-0.0614	-0.8121
0.2232	.518	.505	-0.0984	-0.8000
0.3124	.536	.511	-0.1297	-0.7822
0.4017	.560	.517	-0.1534	-0.7589
0.4910	.589	.525	-0.1680	-0.7303
0.5803	.624	.532	-0.1722	-0.6964
0.6696	.664	.540	-0.1648	-0.6572
0.7589	.710	.548	-0.1449	-0.6124
0.8482	.762	.556	-0.1115	-0.5615
0.9374	.819	.564	-0.0635	-0.5041
1.0267	.881	.571	-0.000	-0.4396

6.5 THE AXI-SYMMETRIC MEMBRANE BEHAVIOR: UNIFORM PRESSURE DISTRIBUTION

If the paraboloidal shell is subjected only to a uniformly distributed pressure q_0, then the equilibrium equations assume the form (see equations 6.4.6, 6.4.7)

$$\gamma \frac{dN_r}{d\gamma} + N_r - N_\theta = 0 , \qquad 6.5.1$$

$$\frac{N_r}{1+(\gamma)^2} + N_\theta = -2f\sqrt{1+(\gamma)^2}\, q_0 \qquad 6.5.2$$

where it should be noted that p_n has been set equal to q_0. The differential equation for N_r becomes (see equation 6.4.13)

$$\frac{dN_r}{d\gamma} + N_r\left[\frac{1}{\gamma} + \frac{1}{\gamma\,[1+(\gamma^2)]}\right] = -\frac{2f\sqrt{1+(\gamma)^2}}{\gamma}\, q_0 \,. \qquad 6.5.3$$

The solution to this ordinary differential equation is

$$N_r = q_0 f\left\{-\sqrt{1+(\gamma)^2} + \frac{\sqrt{1+(\gamma)^2}}{(\gamma)^2}\; C_1\right\}. \qquad 6.5.4$$

The hoop force resultant is obtained by substituting

equation 6.5.4 into equation 6.5.2. There results

$$N_\theta = -q_0 f \left\{ \frac{1+2(\gamma)^2}{\sqrt{1+(\gamma)^2}} + \frac{C_1}{(\gamma)^2\sqrt{1+(\gamma)^2}} \right\}. \qquad 6.5.5$$

The expressions for the strains are

$$\epsilon_r^o = \frac{q_0 f}{Eh} \frac{1}{\sqrt{1+(\gamma)^2}} \left\{ -1+\nu-(2-\nu)(\gamma)^2 - \frac{1+\nu+\nu(\gamma)^2}{(\gamma)^2} C_1 \right\}. \qquad 6.5.6$$

The differential equation for the meridianal displacements becomes (see equation 6.4.19)

$$\frac{du_r^o}{d\gamma} - \frac{u_r^o}{\gamma[1+(\gamma)^2]} = \frac{2f^2 q_0}{Eh} \frac{1}{[1+(\gamma)^2]} \left\{ (2\nu-1)(\gamma)^4 + 2\nu(\gamma)^2 + \left[(\gamma)^2 + 2(1+\nu) + \frac{2(1+\nu)}{(\gamma)^2} \right] C_1 \right\}. \qquad 6.5.7$$

The solution for u_r^o is

$$u_r^o(\gamma) = \frac{2f^2 q_0}{Eh} \left\{ \frac{(2\nu-1)}{3}(\gamma)^3 + \frac{(2\nu+2)\gamma}{3} + C_1\left[\gamma - \frac{(1+\nu)}{\gamma} - \frac{(1+\nu)\gamma}{\sqrt{1+(\gamma)^2}} \ln\left(\frac{1+\sqrt{1+(\gamma)^2}}{\gamma} \right) \right] + \frac{C_2\gamma}{\sqrt{1+(\gamma)^2}} \right\}. \qquad 6.5.8$$

In order to obtain the normal displacement we again make

use of equation 6.4.18. There results

$$w = \frac{2f^2 q_0}{Eh}\left\{\left(\frac{5-\nu}{3}\right)\left(1+\gamma^2\right) + C_1\left(1+\nu\right)\left[1-\frac{1}{(\gamma)^2}-\frac{1}{\sqrt{1+(\gamma)^2}}\right.\right.$$
$$\left.\left.\ell n\left(\frac{1+\sqrt{1+(\gamma)^2}}{\gamma}\right)\right]+\frac{C_2}{\sqrt{1+(\gamma)^2}}\right\}. \qquad 6.5.9$$

For this case wherein the load is a uniform pressure, the non-dimensional force-resultants are defined as

$$N_r^* = \frac{N_r}{2fq_0}\ , \qquad 6.5.10$$

$$N_\theta^* = \frac{N_\theta}{2fq_0} \qquad 6.5.11$$

and the non-dimensional displacements are defined as

$$u_r^* = \frac{Eh u_r^o}{4f^2 q_0}\ , \qquad 6.5.12$$

$$w^* = \frac{Ehw}{4f^2 q_0}\ . \qquad 6.5.13$$

6.5.1 Case 1 BOTH EDGES RESTRAINED IN THE TANGENTIAL DIRECTIONS

The constants for the case of a shell which is restrained at both edges (see section 6.4.1) are as follows:

$$C_1 = \frac{D(\gamma_1)\,B(\gamma_2) - D(\gamma_2)\,B(\gamma_1)}{A(\gamma_1)\,B(\gamma_2) - A(\gamma_2)\,B(\gamma_1)}, \qquad 6.5.1.1$$

$$C_2 = \frac{A(\gamma_1)\,D(\gamma_2) - A(\gamma_2)\,D(\gamma_1)}{A(\gamma_1)\,B(\gamma_2) - A(\gamma_2)\,B(\gamma_1)} \qquad 6.5.1.2$$

where

$$A(\gamma_i) = \gamma_i - \frac{(1+\nu)}{\gamma_i} - \frac{\gamma_i\,(1+\nu)}{\sqrt{1+(\gamma_i)^2}}\,\ln\left(\frac{1+\sqrt{1+(\gamma_i)^2}}{\gamma_i}\right), \qquad 6.5.1.3$$

$$B(\gamma_i) = \frac{\gamma_i}{\sqrt{1+(\gamma_i)^2}}, \qquad 6.5.1.4$$

$$D(\gamma_i) = -\frac{(2\nu-1)}{3}\,(\gamma_i)^3 - \frac{(2\nu+2)}{3}\,\gamma_i \qquad 6.5.1.5$$

and γ_1, γ_2 are the boundaries which are restrained. Again the determinant Δ,

$$\Delta = A(\gamma_1)\,B(\gamma_2) - A(\gamma_2)\,B(\gamma_1) \qquad 6.5.1.6$$

does not vanish for all positive γ_1, γ_2 ($\gamma_2 > \gamma_1$).

Six different sizes of shells have been analyzed in this section. It is interesting to note that the largest displacement occurs at the outer boundary. Again, the explanation lies in the restricted type of boundary condition which accompanies membrane behavior (see Section 6.4.1 and Figure 6.4.1). The outer portion of the shell, in order to preserve $u_r = 0$ at the boundary, must open up somewhat like a flower in order to accommodate the strains developed by the loads. It is this "opening-up" which induces the larger w^* displacement.

The six cases are summarized in the following table:

Table 6.5.1.1

Case	Restrained at γ_1	Restrained at γ_2
6.5.1.1	.0446	.6250
6.5.1.2	.0104	.3854
6.5.1.3	.3854	.6249
6.5.1.4	.0104	.3854
6.5.1.5	.0446	.8035
6.5.1.6	.0446	1.0267

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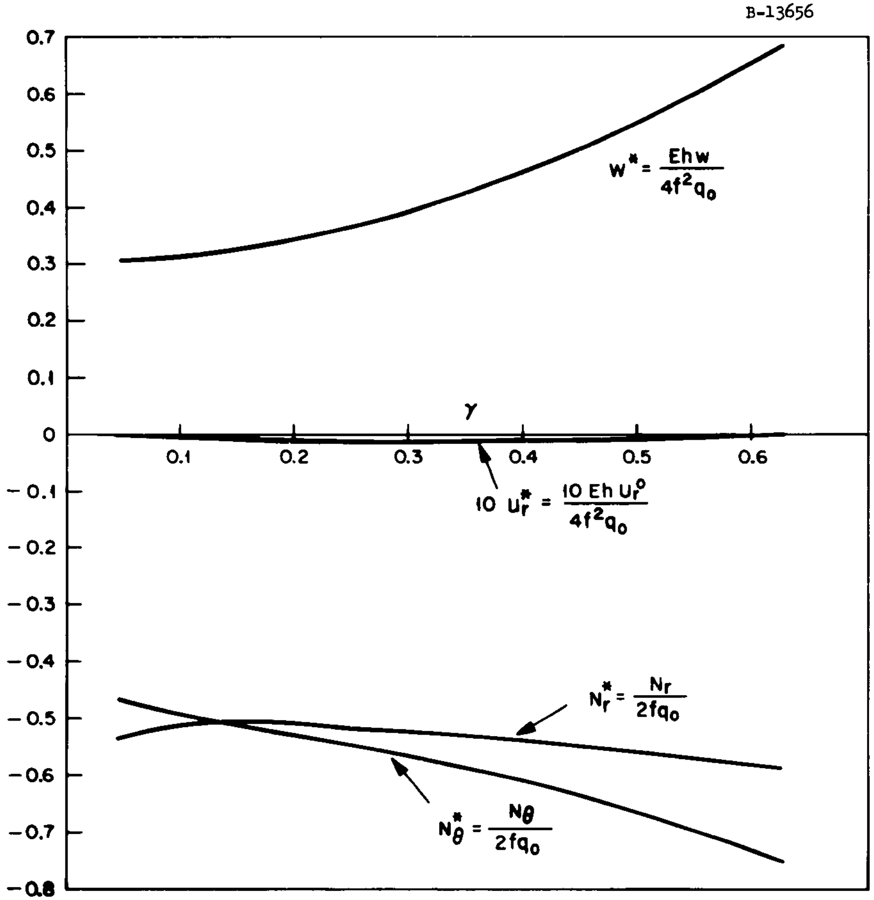

CASE 6.5.1.1 UNIFORM PRESSURE
RESTRAINED $\gamma_1 = .0446$
RESTRAINED $\gamma_2 = .625$

γ	N_r^*	N_θ^*	u^*	w^*
0.0446	− .536	−.466	−.000	.305
0.1306	− .508	−.508	−.00506	.320
0.2166	− .513	−.533	−.00810	.350
0.3025	− .523	−.565	−.00958	.395
0.3885	− .537	−.606	−.0093	.454
0.4745	− .554	−.655	−.0073	.526
0.5605	− .573	−.710	−.00366	.610
0.6249	− .590	−.755	.000	.682

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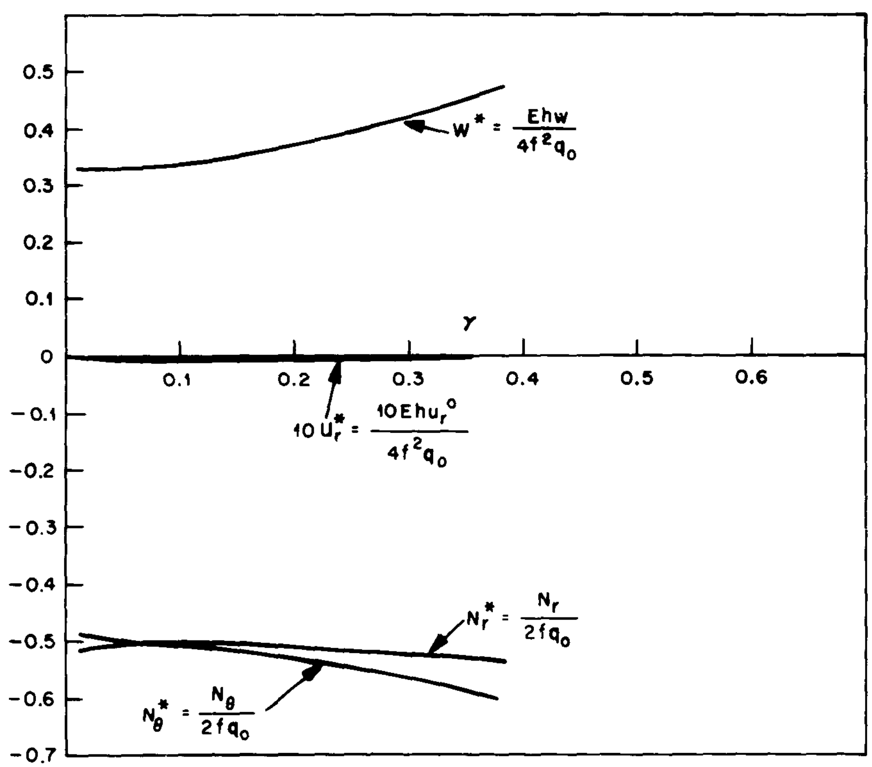
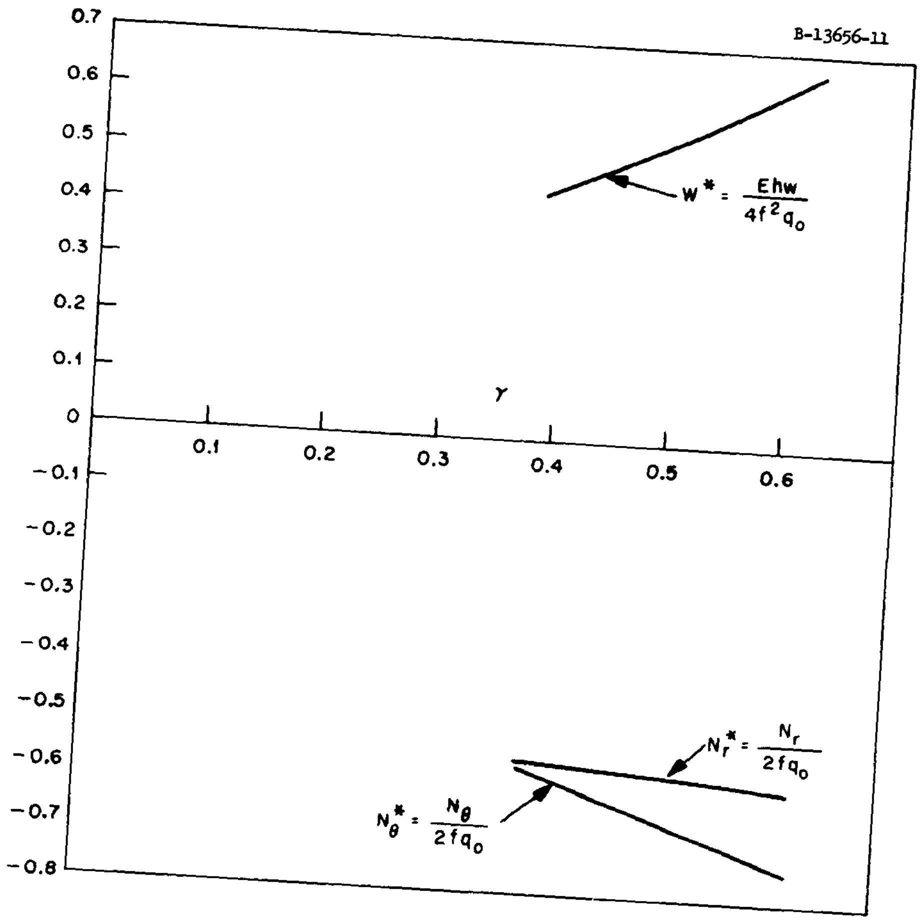


CASE 6.5.1.2 UNIFORM PRESSURE

RESTRAINED γ_1 = .0104

RESTRAINED γ_2 = .3854

γ	N_r^*	N_θ^*	u^*	w^*
0.0104	−.516	−.484	.000	.330
0.0520	−.501	−.501	−.00100	.332
0.0937	−.502	−.506	−.00176	.338
0.1354	−.505	−.514	−.00236	.348
0.1770	−.508	−.523	−.00275	.361
0.2187	−.512	−.535	−.00287	.378
0.2604	−.517	−.549	−.00269	.397
0.3020	−.522	−.566	−.00216	.420
0.3437	−.529	−.584	−.00127	.447
0.3854	−.536	−.605	.000	.476



CASE 6.5.1.3 UNIFORM PRESSURE
RESTRAINED $\gamma_1 = .3854$
RESTRAINED $\gamma_2 = .6249$

γ	N_r^*	N_θ^*	U^*	w^*
0.3854	−.567	−.578	.000	.437
0.4270	−.569	−.606	−.00136	.470
0.4687	−.574	−.634	−.00206	.506
0.5104	−.580	−.662	−.00217	.544
0.5520	−.587	−.692	−.00177	.586
0.5937	−.596	−.722	−.000908	.631
0.6249	−.603	−.746	.000	.666

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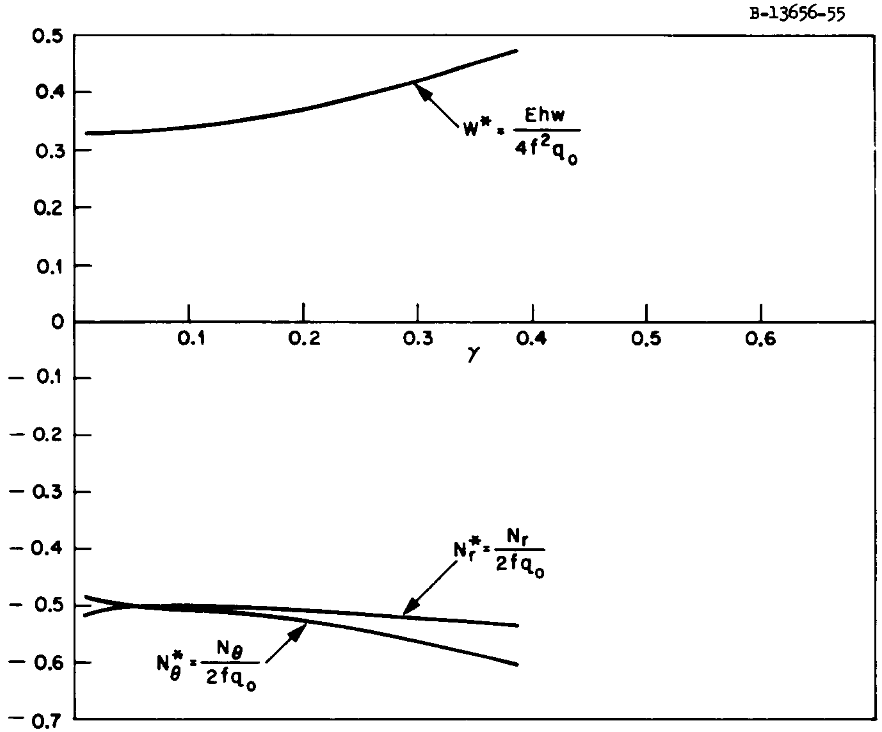

Case 6.5.1.4 Uniform Pressure
Restrained γ_1 = 0.0104
Restrained γ_2 = 0.3854

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0104	-.515	-.484	.000	.329
0.0312	-.501	-.498	-.0005	.330
0.0520	-.501	-.501	-.001	.332
0.0833	-.501	-.504	-.0015	.336
0.1249	-.504	-.511	-.0022	.345
0.1666	-.506	-.520	-.0026	.357
0.2083	-.510	-.531	-.0028	.373
0.2499	-.515	-.545	-.0027	.392
0.2916	-.520	-.561	-.0023	.414
0.3333	-.527	-.579	-.0015	.439
0.3749	-.534	-.599	-.0003	.468
0.3854	-.535	-.605	.000	.476

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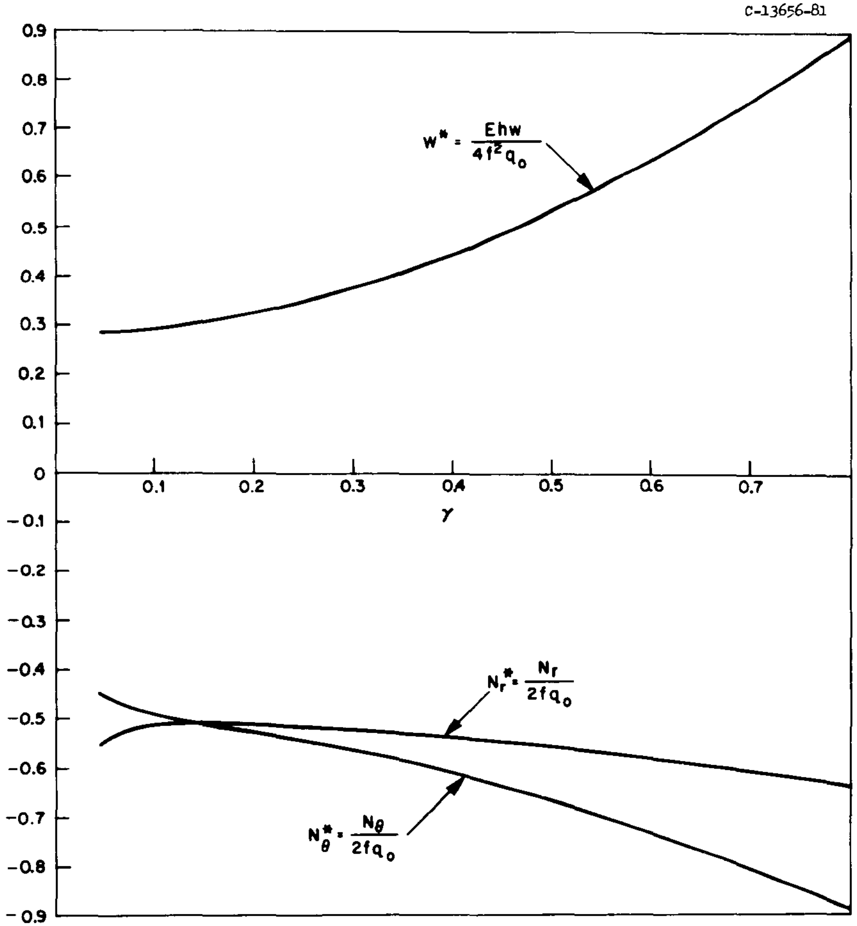
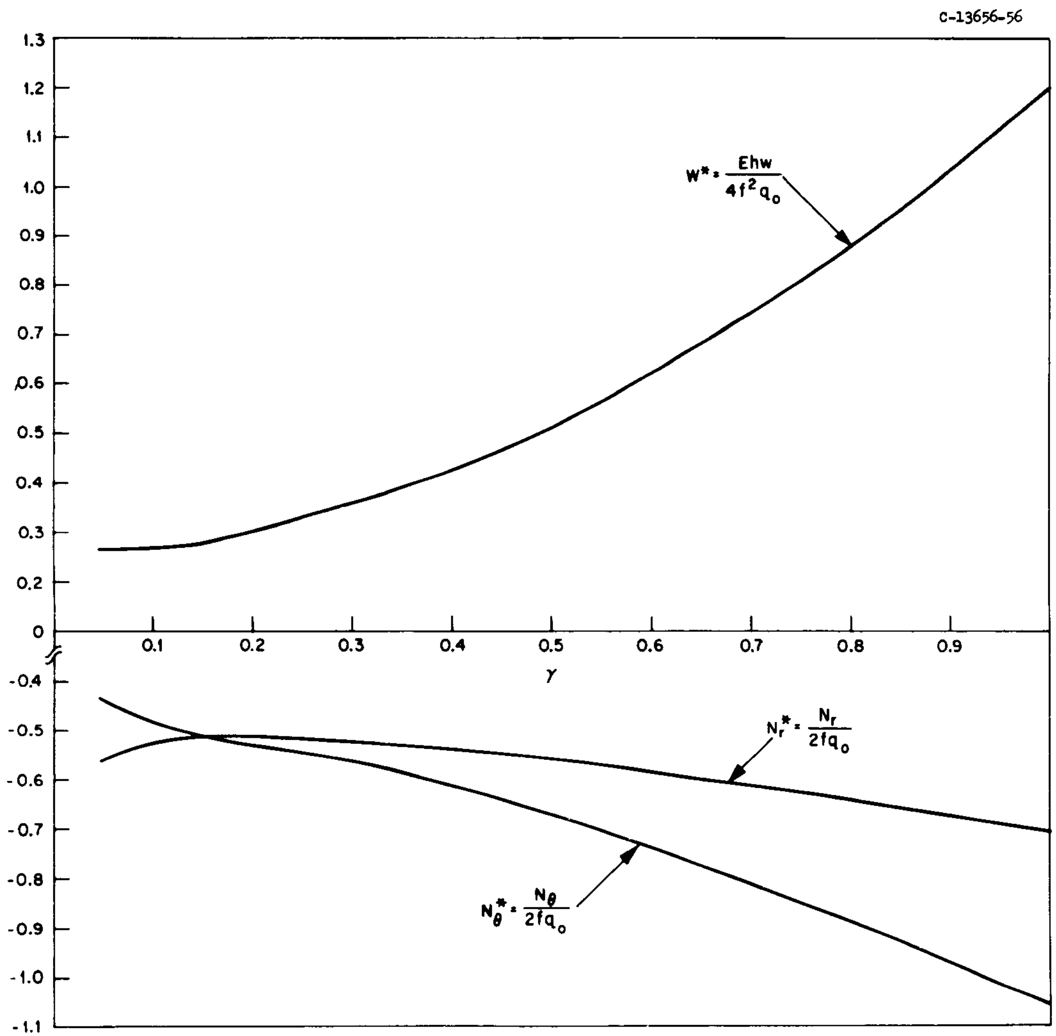


Case 6.5.1.5 Uniform Pressure
Restrained γ_1 = 0.0446
Restrained γ_2 = 0.8035

γ	N_r^*	N_θ^*	u_r^*	w^*
0.0446	-.551	-.450	.000	.284
0.1339	-.510	-.507	-.007	.301
0.2232	-.514	-.534	-.012	.333
0.3124	-.524	-.569	-.015	.381
0.4017	0.539	-.613	-.016	.445
0.4910	-.557	-.664	-.015	.522
0.5803	-.578	-.723	-.012	.613
0.6696	-.602	-.787	-.0083	.718
0.7589	-.627	-.856	-.0029	.835
0.8035	-.641	-.892	-0.	.898



Case 6.5.1.6 Uniform Pressure
Restrained γ_1 = 0.0446
Restrained γ_2 = 1.0267

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0446	-.566	-.435	.000	.265
0.1339	-.511	-.506	-.0098	.281
0.2232	-.515	-.534	-.0165	.314
0.3124	-.525	-.569	-.0212	.363
0.4017	-.539	-.612	-.0238	.426
0.4910	-.557	-.664	-.0241	.505
0.5803	-.578	-.723	-.0024	.596
0.6696	-.602	-.787	-.0191	.702
0.7589	-.627	-.856	-.0147	.819
0.8482	-.655	-.929	-.0096	.949
0.9374	-.685	-1.005	-.0045	1.092
1.0267	-.716	-1.084	.000	1.245

6.5.2 ONE EDGE FREE AND ONE EDGE RESTRAINED IN THE TANGENTIAL DIRECTION

We will specify the free edge to be γ_1, and γ_2 to be the restrained edge (see section 6.4.2). Then the constants are as follows:

$$C_1 = (\gamma_1)^2, \qquad 6.5.2.1$$

$$C_2 = \frac{\sqrt{1+(\gamma_2)^2}}{\gamma_2}\left\{\left[\frac{(1+\nu)\gamma_2}{\sqrt{1+(\gamma_2)^2}}\,\ln\left(\frac{1+\sqrt{1+(\gamma_2)^2}}{\gamma_2}\right)+\frac{(1+\nu)}{\gamma_2}-\gamma_2\right]C_1 - \left[\frac{(2\nu-1)}{3}(\gamma_2)^3+\frac{2(1+\nu)}{3}\gamma_2\right]\right\}. \qquad 6.5.2.2$$

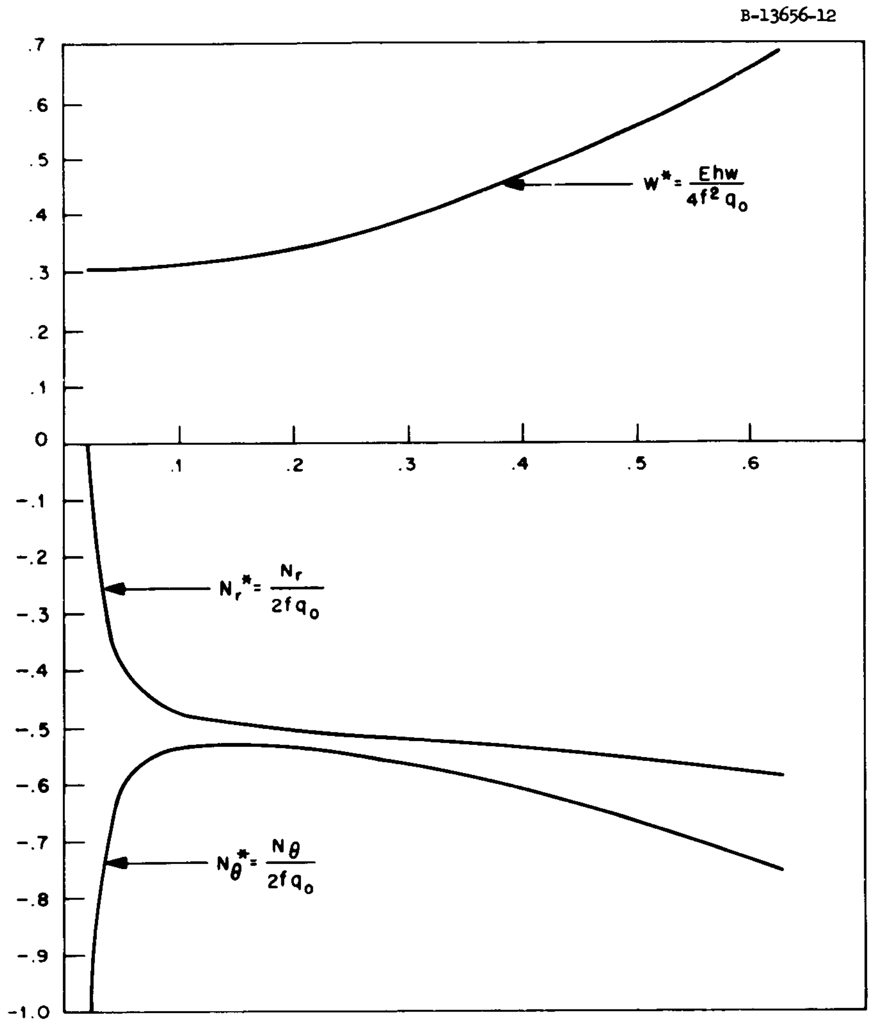
Twenty-three different size shells have been analyzed in this section. A small hole, while affecting the distribution of N_r^* and N_θ^*, does not significantly affect the displacement W^* (compare cases 6.4.1.1 and 6.5.2.5). The discussion of W^* in the previous section also applies for the cases in this section where the free boundary is the inner radius. For the cases where the shell is supported at the inner radius and is free at the outer boundary (see e.g., case 6.5.2.12), the W^* displacements are larger at the support. Again, this is somewhat contrary to one's intuition. The explanation is similar to that advanced in the previous sections; the larger strains generated at the supported boundary can be accommodated with the restriction of $u_r = 0$ only by large values

of the normal displacement. Note also (see case 6.4.2.12) that the tangential displacement becomes appreciable if the free boundary is the outer radius of the shell.

A list summarizing the cases which have been studied is contained in Table 6.5.2.1.

Table 6.5.2.1

Case	Free at γ_1	Restrained at γ_2
6.5.2.1	.0223	.6249
6.5.2.2	.0892	.6250
6.5.2.3	.1339	.6250
6.5.2.4	.1785	.6250
6.5.2.5	.0446	.6250
6.5.2.6	.0104	.3854
6.5.2.7	.6250	.3854
6.5.2.8	.1785	.8035
6.5.2.9	.4017	.8035
6.5.2.10	.1785	1.0267
6.5.2.11	.4017	1.0267
6.5.2.12	.8020	.2083
6.5.2.13	.8020	.3854
6.5.2.14	.8020	.4687
6.5.2.15	.9990	.3854
6.5.2.16	1.0000	.4687
6.5.2.17	.0104	.1041
6.5.2.18	.6250	.2083
6.5.2.19	.0104	.2083
6.5.2.20	.0104	.4687
6.5.2.21	.0104	.8020
6.5.2.22	.0104	.9990
6.5.2.23	.6250	.0520

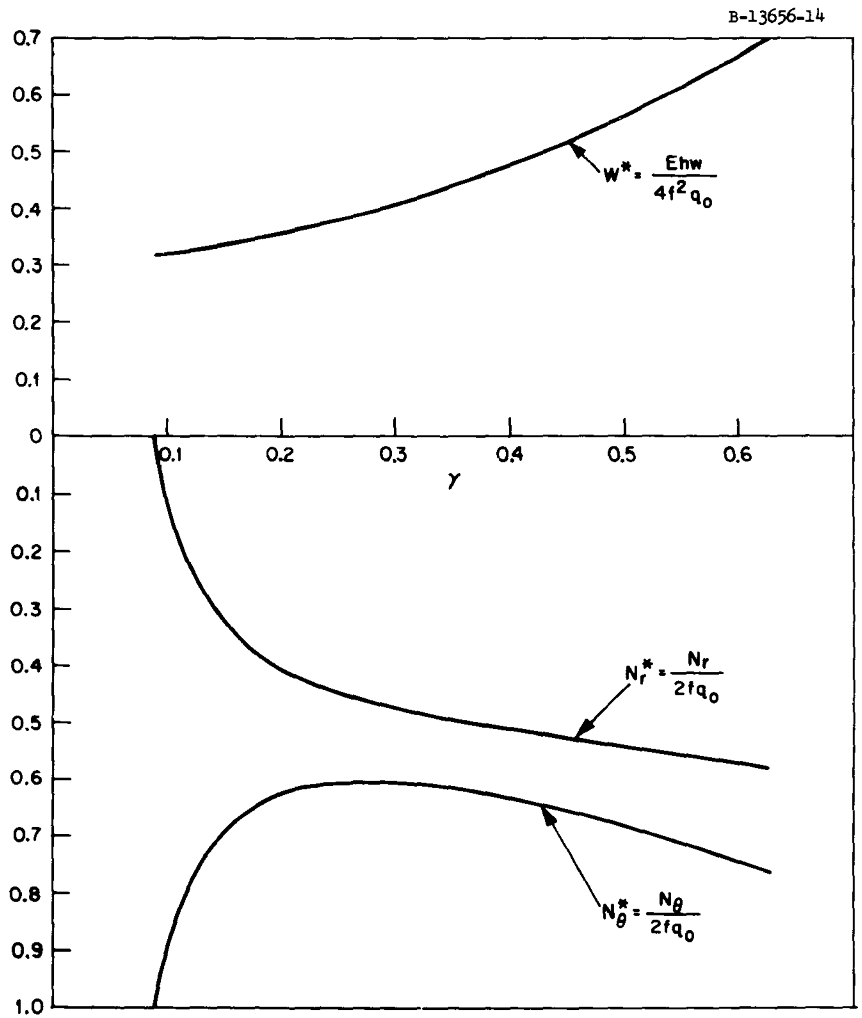


CASE 6.5.2.1 UNIFORM PRESSURE

FREE γ_1 =.0223

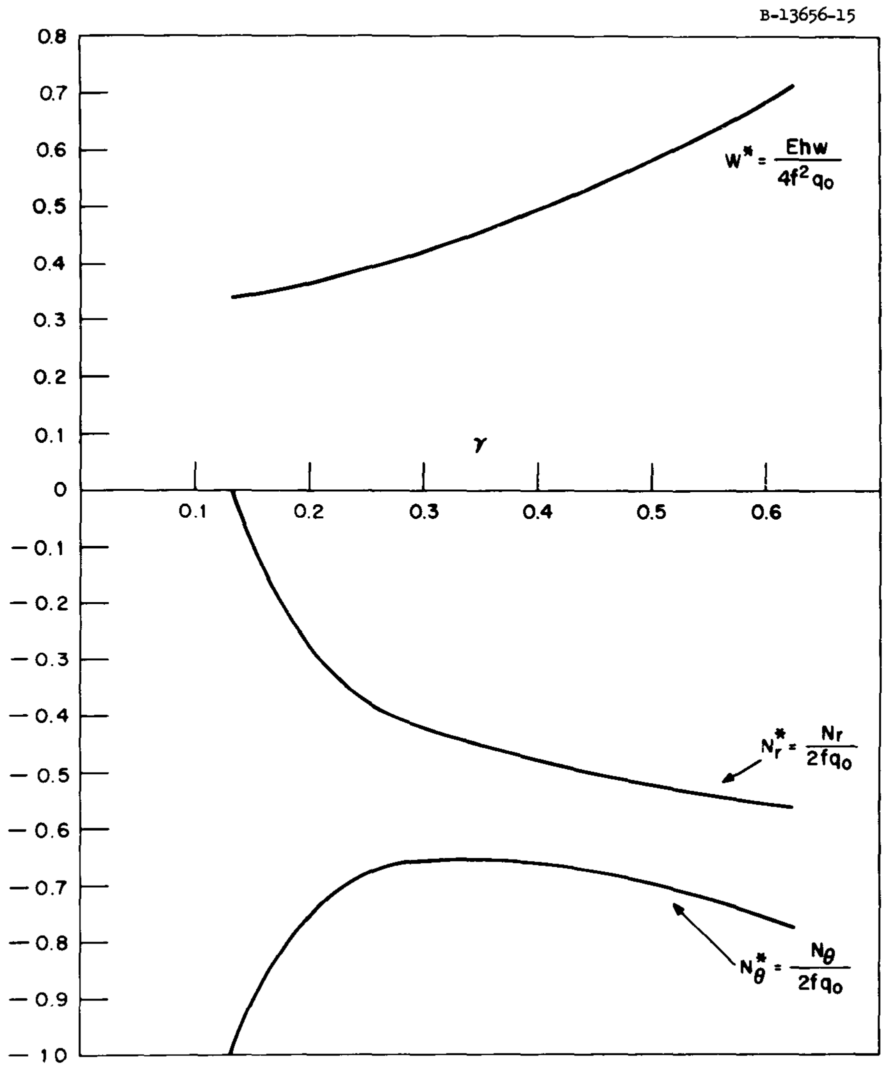
RESTRAINED γ_2 =.6249

γ	N_r^*	N_θ^*	u^*	w^*
.0223	-.000	-1.00	-.0156	.304
.0446	-.375	- .626	-.009	.305
.1339	-.490	- .527	-.008	.322
.2232	-.507	- .541	-.010	.354
.3124	-.521	- .573	-.011	.402
.4017	-.537	- .615	-.0098	.465
.4910	-.556	- .666	-.0071	.542
.5803	-.577	- .724	-.0027	.633
.6249	-.586	- .756	.000	.683



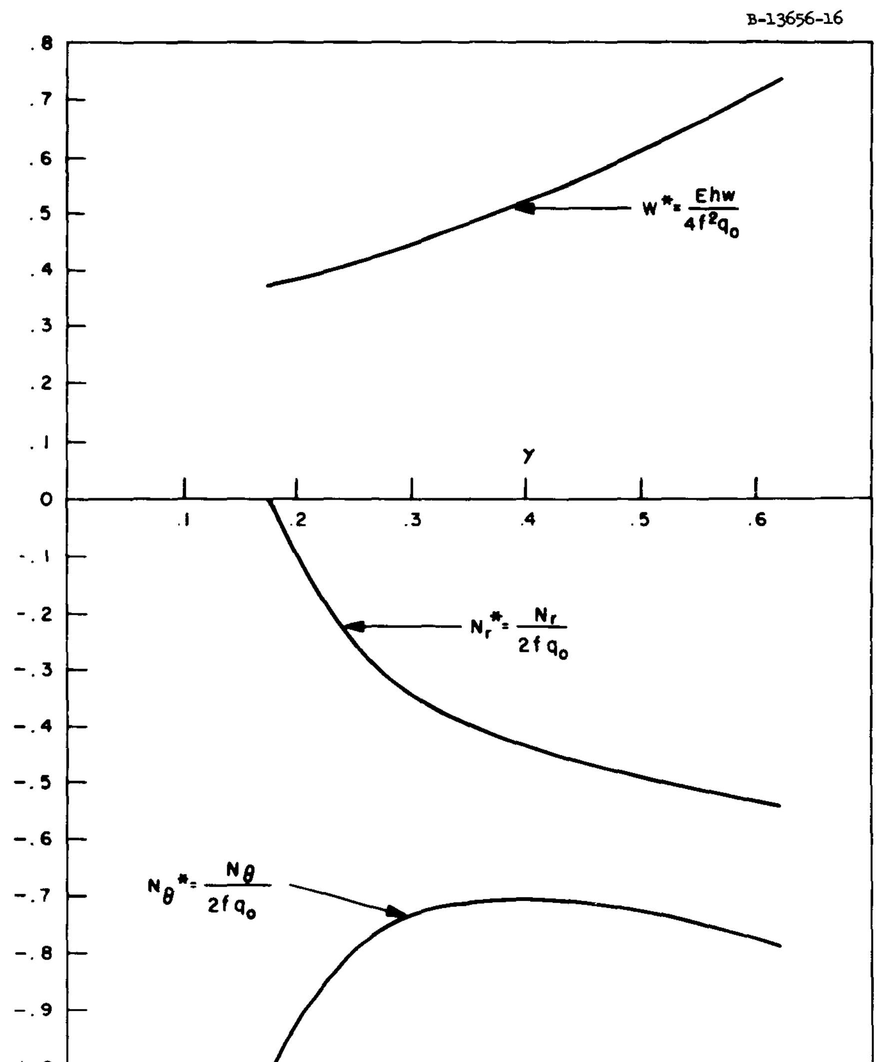
CASE 6.5.2.2 UNIFORM PRESSURE
FREE γ =.0892
RESTRAINED γ =.625

γ	N_r^*	N_θ^*	U_r^*	W^*
.0892	-.000	-1.00	-.0616	.318
.1098	-.171	-.837	-.0514	.323
.1923	-.399	-.633	±.033	.351
.2747	-.464	-.606	-.0255	.391
.3571	-.498	-.620	-.0201	.444
.4345	-.524	-.653	-.0147	.509
.5219	-.548	-.698	-.0087	.585
.5837	-.565	-.736	-.0036	.650
.6249	-.578	-.764	-.000	.696



CASE 6.5.2.3 UNIFORM PRESSURE
FREE γ = .1339
RESTRAINED γ = .625

γ	N_r^*	N_θ^*	U_r^*	W^*
.1339	− .000	−1.00	−.091	.340
.1535	− .121	− .893	−.080	.348
.2321	− .342	− .702	−.054	.383
.3107	− .426	− .658	−.0395	.428
.3892	− .473	− .662	−.0288	.484
.4678	− .507	− .688	−.0192	.551
.5464	− .535	− .727	−.0097	.628
.6249	− .562	− .775	−.000	.715

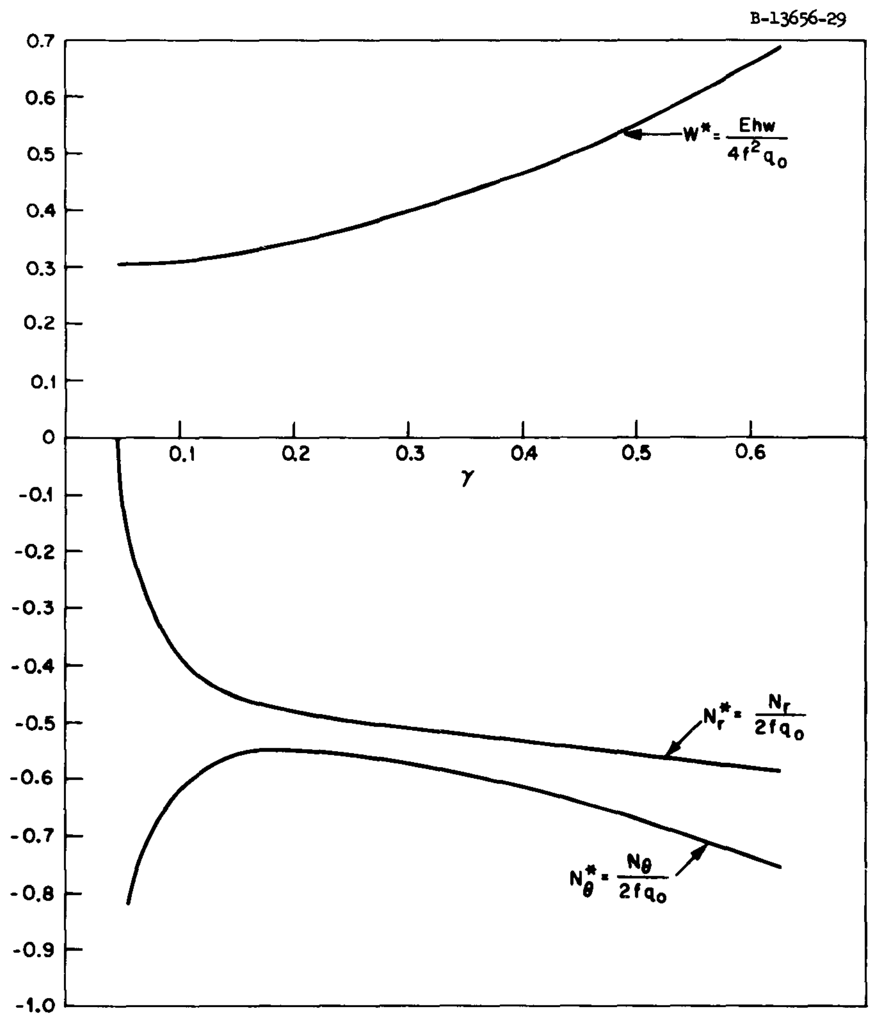


CASE 6.5.2.4 UNIFORM PRESSURE

FREE $\gamma = .1785$

RESTRAINED $\gamma = .625$

γ	N_r^*	N_θ^*	U^*	W^*
.1785	-.000	-1.016	-.117	.375
.1971	-.0916	- .931	-.106	.384
.2715	-.294	- .762	-.074	.425
.3459	-.388	- .712	-.0536	.475
.4203	-.444	- .707	-.037	.533
.4947	-.485	- .726	-.023	.601
.5691	-.519	- .759	-.0097	.677
.6249	-.541	- .790	.000	.740



Case 6.5.2.5 Uniform Pressure
Free $\gamma_1 = 0.0446$
Restrained $\gamma_2 = 0.625$

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0446	-.000	-1.00	-.031	.306
0.0661	-.273	-.730	-.022	.309
0.1521	-.462	-.560	-.0148	.329
0.2380	-.496	-.559	-.0138	.363
0.3240	-.516	-.585	-.0130	.412
0.4100	-.534	-.624	-.011	.474
0.4960	-.554	-.672	-.0076	.549
0.5820	-.575	-.727	-.0028	.637
0.6249	-.587	-.757	.000	.686

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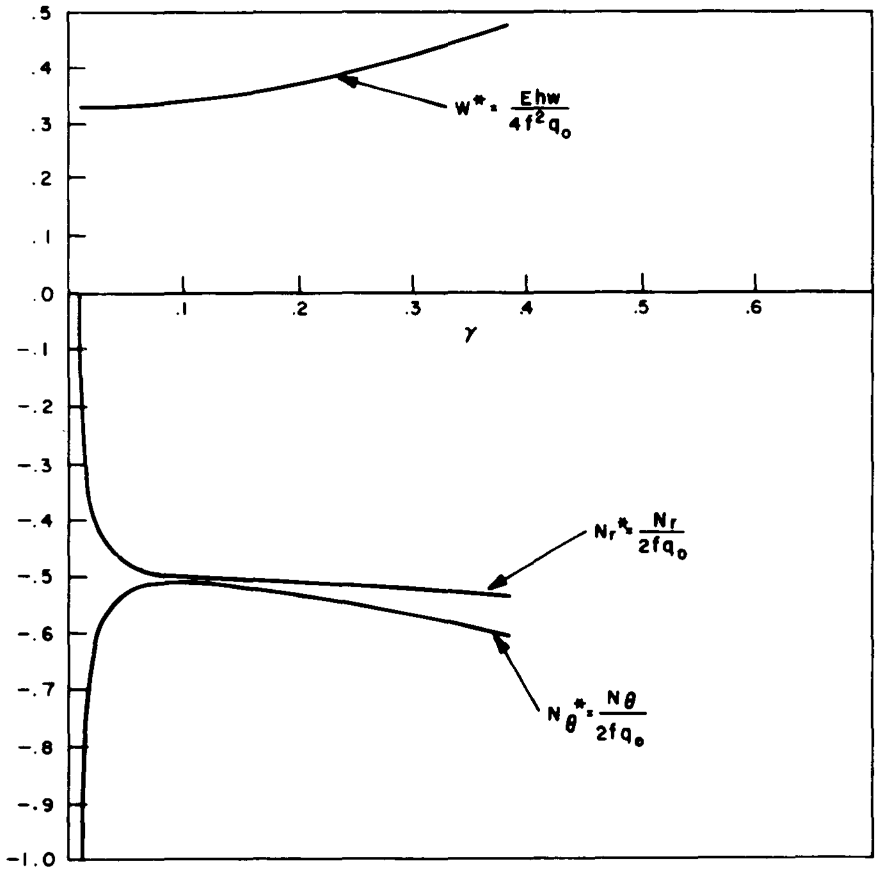

CASE 6.5.2.6 UNIFORM PRESSURE

FREE γ = .0104
RESTRAINED γ = .3854

γ	N_r^*	N_θ^*	u^*	w^*
.0104	-.000	-1.00	-.00698	.330
.0208	-.375	- .625	-.00380	.330
.0624	-.487	- .517	-.00234	.334
.1041	-.498	- .513	-.00259	.341
.1458	-.503	- .518	-.00292	.351
.1874	-.507	- .528	-.00311	.365
.2291	-.512	- .540	-.00306	.383
.2708	-.517	- .554	-.00272	.403
.3124	-.523	- .571	-.00205	.427
.3541	-.530	- .590	-.00102	.454
.3854	-.535	- .605	.000	.477

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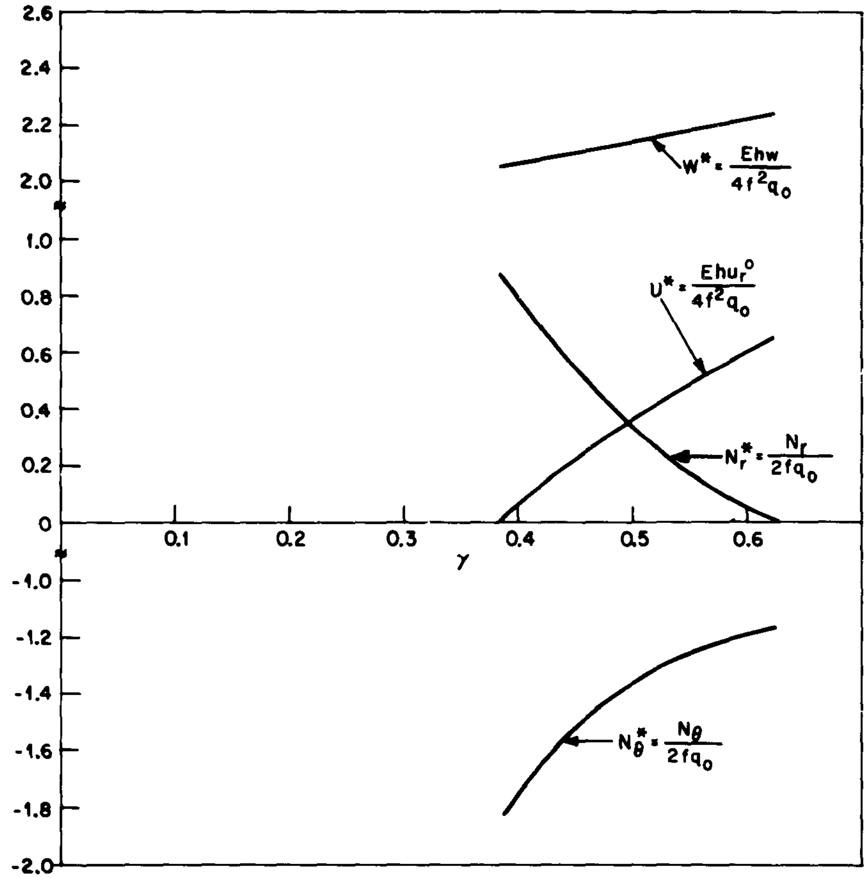
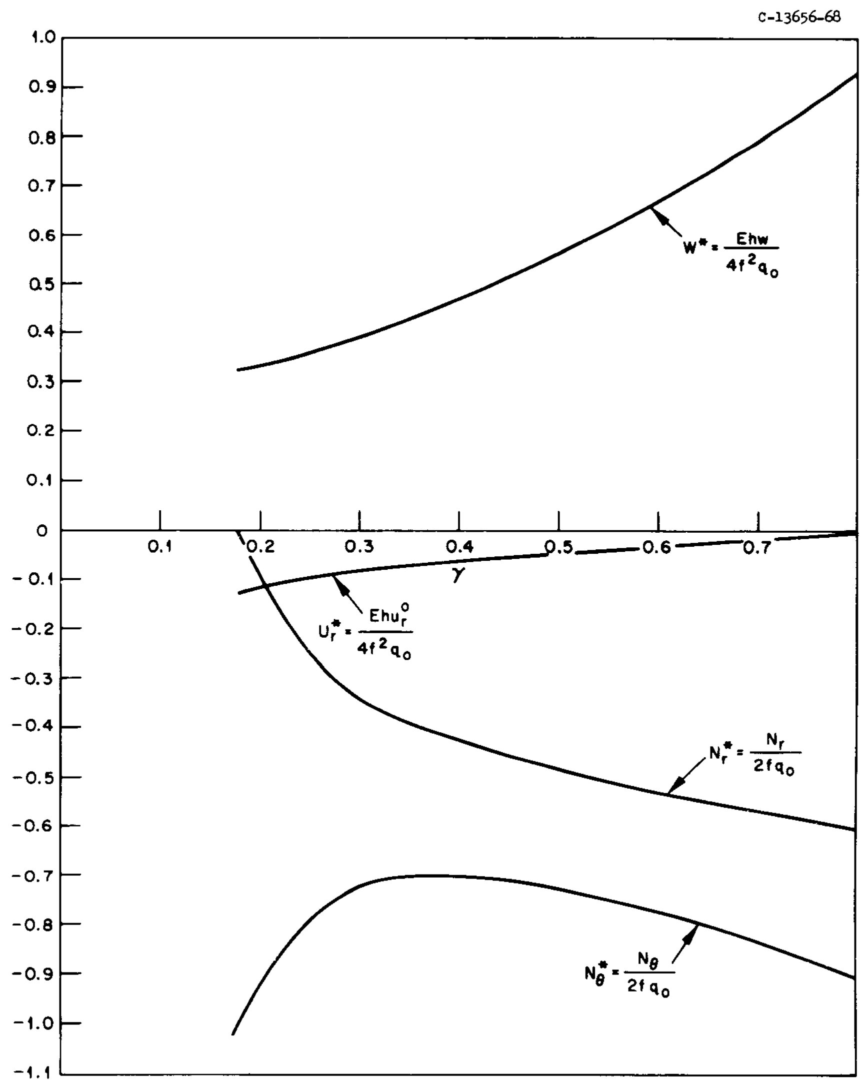


Case 6.5.2.7 Uniform Pressure
Free $\gamma_1 = 0.3854$
Restrained $\gamma_2 = 0.625$

γ	N_r^*	N_θ^*	U_r^*	W^*
0.3854	.873	-1.832	-.000	2.24
0.4270	.621	-1.612	.137	2.277
0.4687	.429	-1.456	.262	2.309
0.5104	.280	-1.345	.376	2.340
0.5520	.161	-1.266	.481	2.372
0.5937	.0628	-1.209	.579	2.403
0.6249	.00008	-1.179	.648	2.428



Case 6.5.2.8 Uniform Pressure
Free γ_1 = 0.1785
Restrained γ_2 = 0.8035

γ	N_r^*	N_θ^*	U_r^*	W^*
0.1785	-0.	-1.015	-.125	.328
0.2678	-.387	-.766	-.0877	.376
0.3571	-.398	-.708	-.0669	.438
0.4464	-.459	-.711	-.0516	.512
0.5357	-.504	-.742	-.0382	.599
0.6259	-.541	-.789	-.0252	.699
0.7142	-.576	-.847	-.0125	.811
0.8035	-.609	-.912	.000	.935

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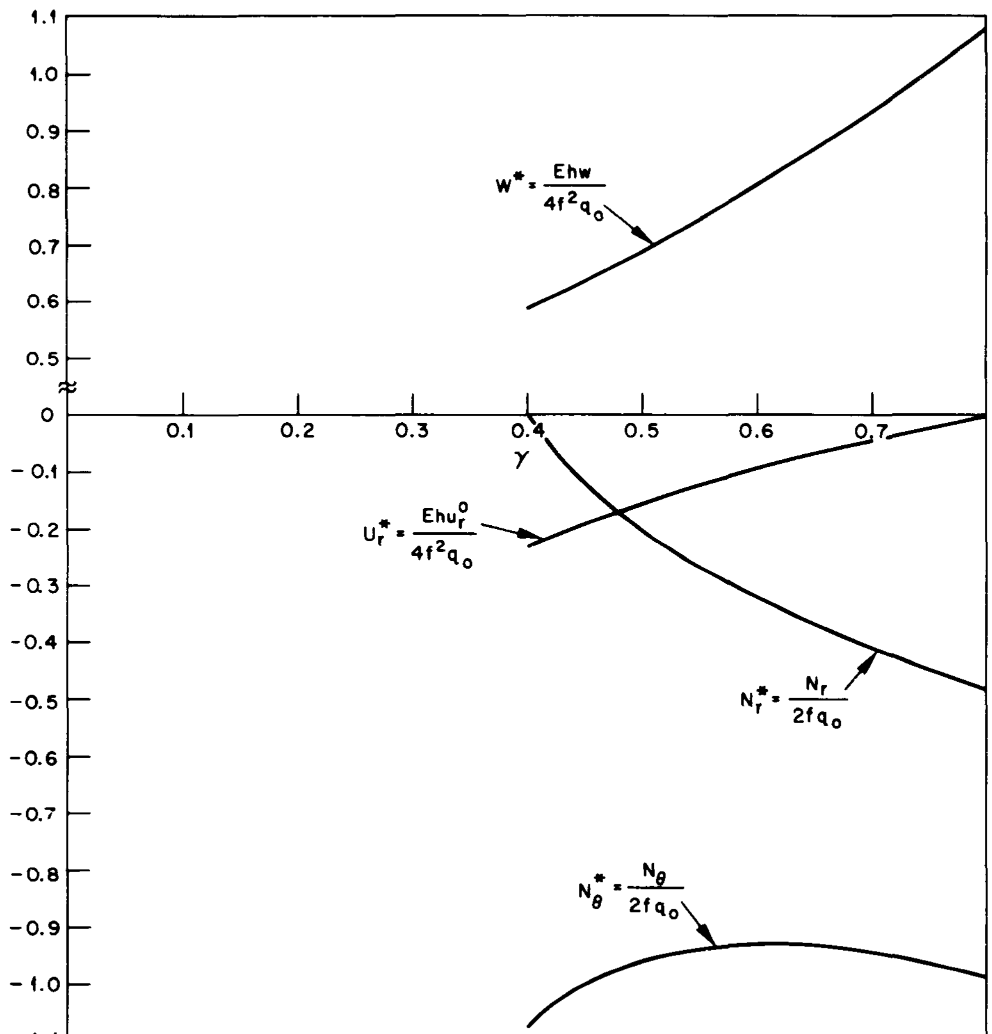
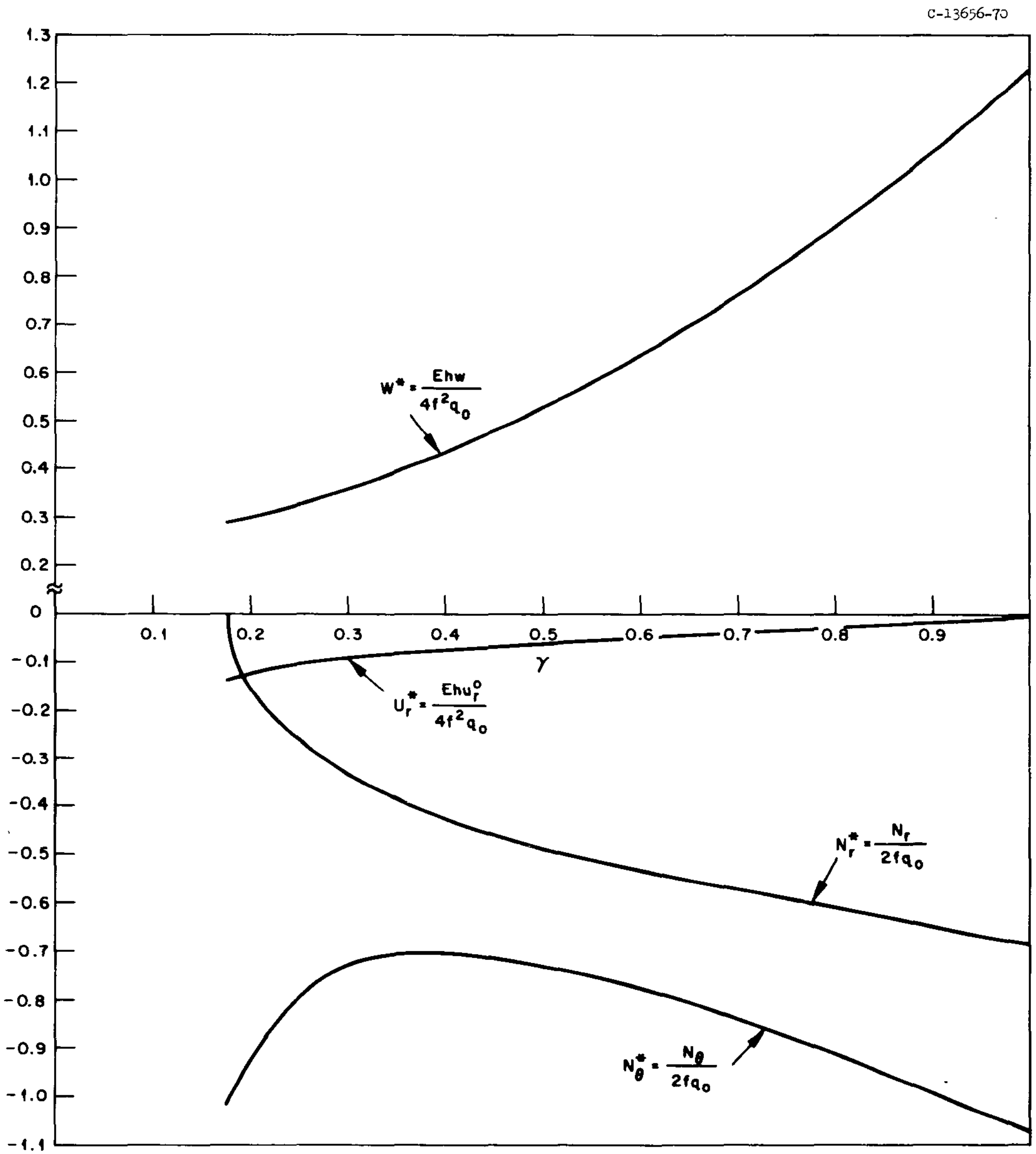


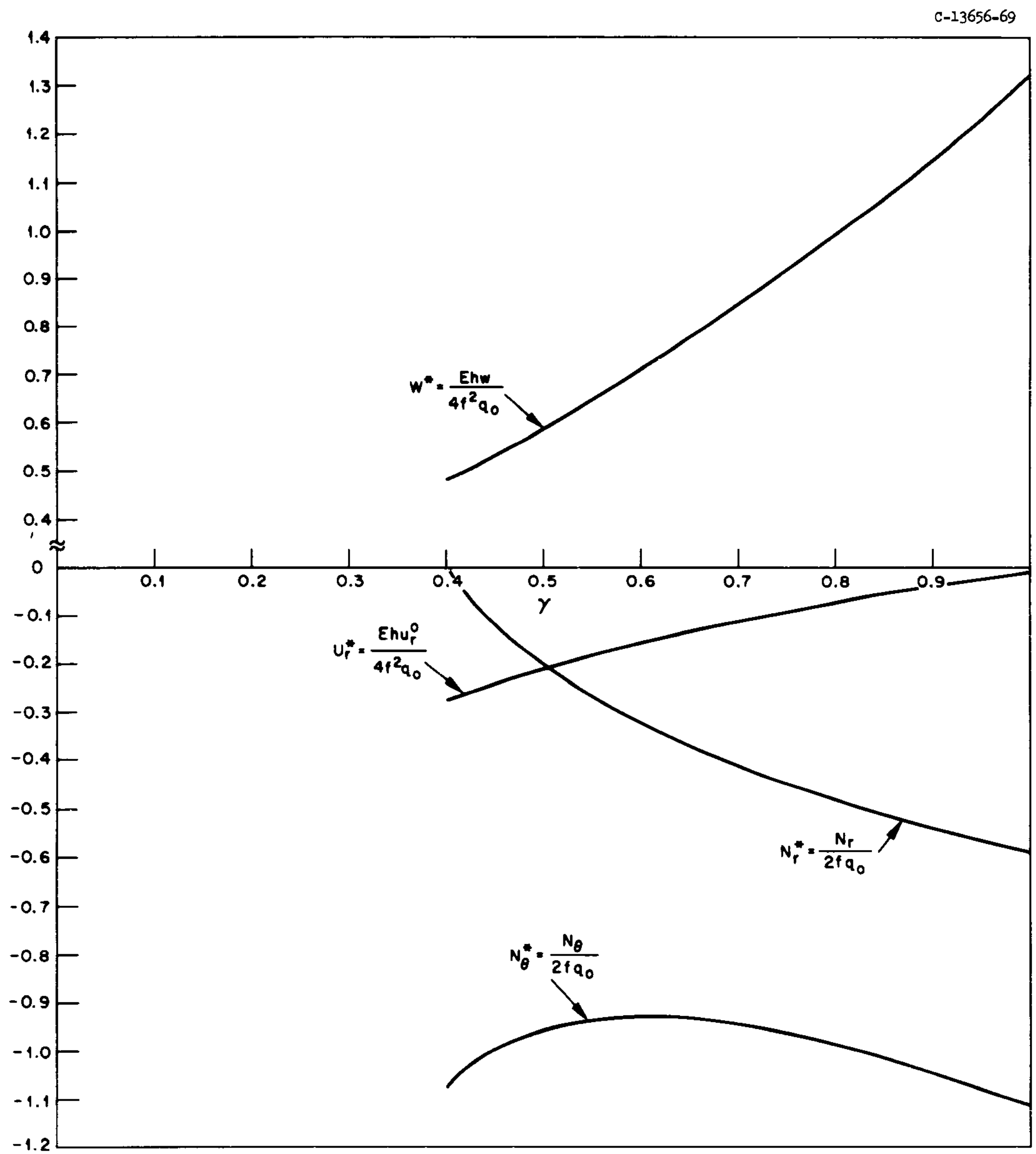
Case 6.5.2.9 Uniform Pressure
Free $\gamma_1 = 0.4017$
Restrained $\gamma_2 = 0.8035$

γ	N_r^*	N_θ^*	U_r^*	W^*
0.4017	-0.	-1.077	-0.22	.589
0.4910	-.184	-.965	-.162	.682
0.5803	-.301	-.931	-.108	.785
0.6696	-.385	-.937	-.0612	.897
0.7589	-.451	-.968	-.0193	1.020
0.8035	-0.481	-.990	-0.	1.085



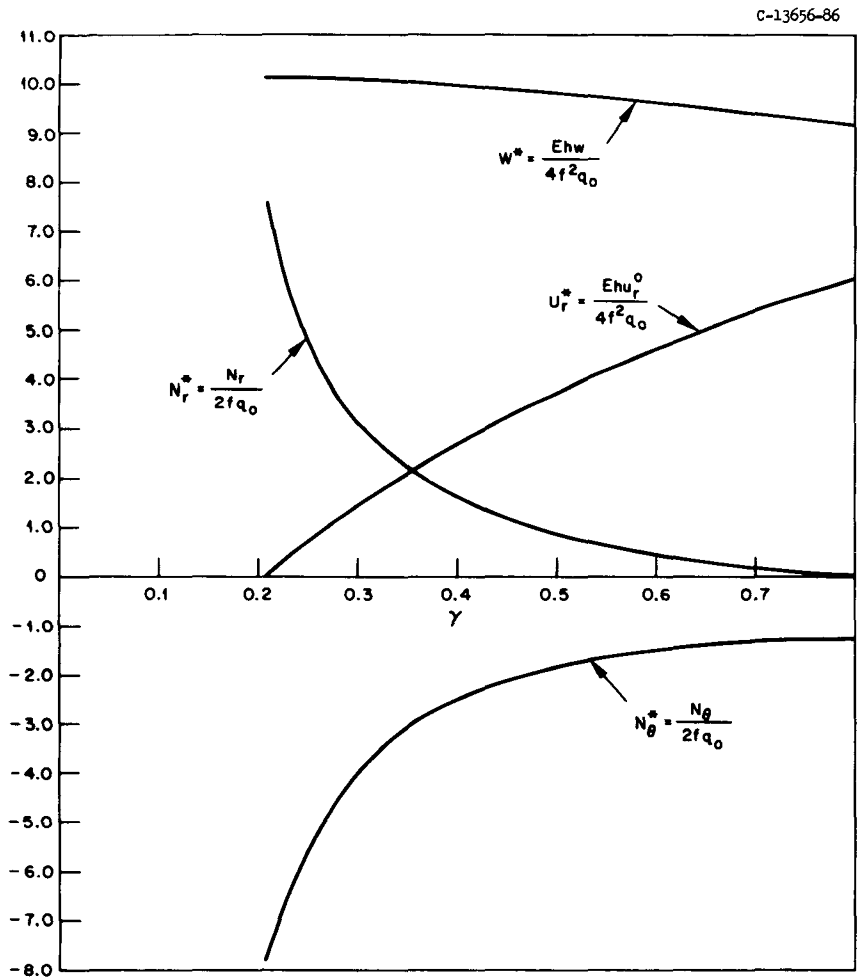
Case 6.5.2.10 Uniform Pressure
Free γ_1 = 0.1785
Restrained γ_2 = 1.0267

γ	N_r^*	N_θ^*	U_r^*	W^*
0.1785	-0.	-1.015	-.132	.290
0.2678	-.287	-.766	-.0978	.339
0.3571	-.398	-.708	-.0800	.401
0.4464	-.459	-.711	-.0675	.476
0.5357	-.504	-.742	-.0565	.565
0.6249	-.541	-.789	-.0458	.566
0.7142	-.576	-.847	-.0350	.779
0.8035	-.609	-.912	-.0243	.905
0.8928	-.643	-.982	-.0139	1.042
0.9821	-.677	-1.056	-.0043	1.191
1.0267	-.694	-1.094	.000	1.270



Case 6.5.2.1i Uniform Pressure
Free γ_1 =0.4017
Restrained γ_2 =1.0267

γ	N_r^*	N_θ^*	U_r^*	W^*
0.4017	-0.	-1.077	-.273	.480
0.491	-.184	-.965	-.214	.578
0.580	-.301	-.931	-.166	.684
0.669	-.385	-.937	-.126	.800
0.758	-.451	-.968	-.089	.927
0.848	-.508	-.015	-.056	1.064
0.937	-.559	-1.072	-.0270	1.211
1.0267	-.606	-1.137	.000	1.369



Case 6.5.2.12 Uniform Pressure
Free γ_1 = 0.8020
Restrained γ_2 = 0.2083

γ	N_r^*	N_θ^*	U_r^*	W^*
0.2083	7.059	-7.787	-.000	10.118
0.2604	4.384	-5.139	.899	10.122
0.3124	2.927	-3.714	1.651	10.094
0.3645	2.043	-2.868	2.310	10.041
0.4166	1.465	-2.332	2.902	9.968
0.4687	1.064	-1.977	3.442	9.880
0.5208	.773	-1.735	3.939	9.782
0.5729	.553	-1.568	4.397	9.675
0.6249	.381	-1.453	4.822	9.564
0.6770	.234	-1.374	5.215	9.451
0.7291	.129	-1.322	5.580	9.338
0.7812	.0342	-1.290	5.919	9.227
0.8020	.000	-1.281	6.047	9.183

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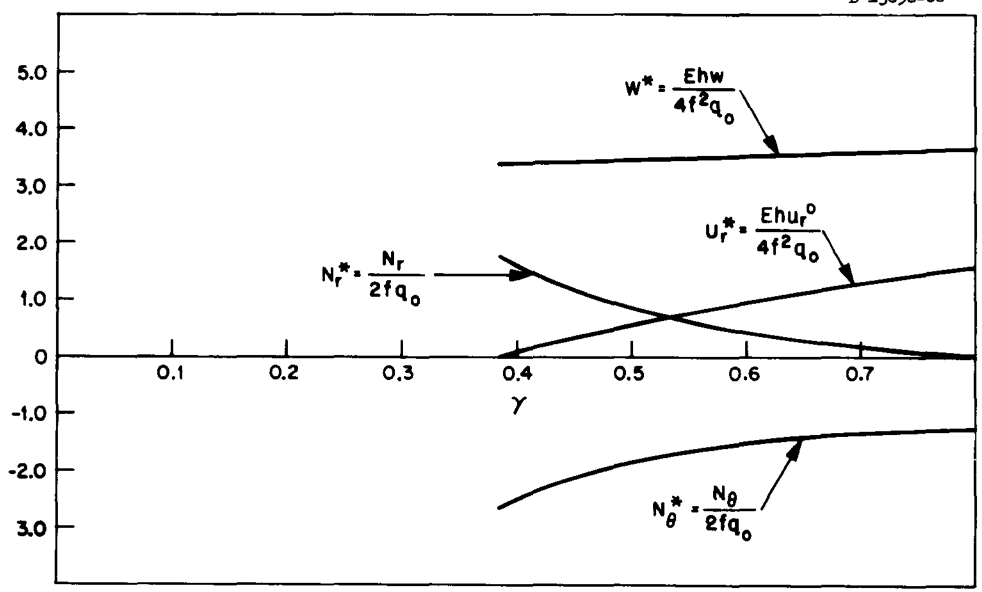
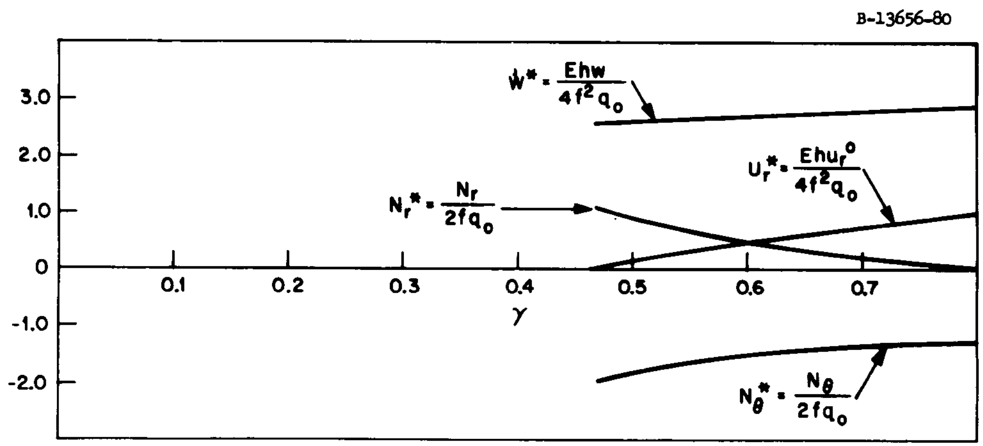


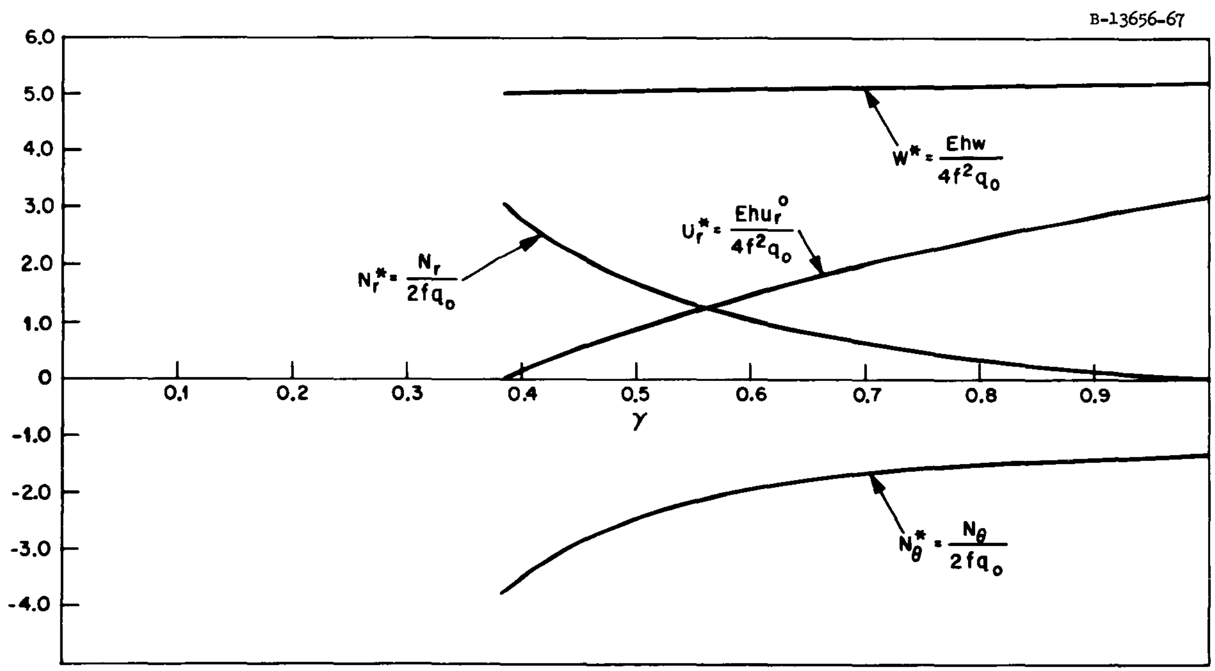
Case 6.5.2.13 Uniform Pressure
Free γ_1 = 0.8020
Restrained γ_2 = 0.3854

γ	N_r^*	N_θ^*	U_r^*	w^*
0.3854	1.784	-2.625	-.000	3.387
0.4270	1.373	-2.249	.225	3.421
0.4687	1.064	-1.977	.428	3.450
0.5104	.824	-1.777	.614	3.477
0.5520	.634	-1.628	.786	3.501
0.5937	.479	-1.517	.945	3.525
0.6354	.351	-1.435	1.094	3.547
0.6770	.243	-1.374	1.233	3.570
0.7187	.151	-1.331	1.364	3.593
0.7604	.0707	-1.301	1.488	3.618
0.8020	-0.	-1.281	1.604	3.643



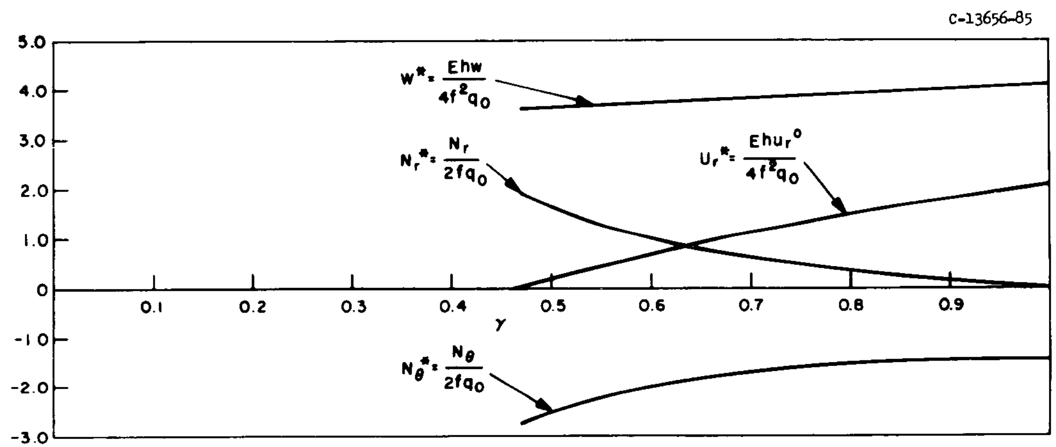
Case 6.5.2.14 Uniform Pressure
Free $\gamma_1 = 0.8020$
Restrained $\gamma_2 = 0.4687$

γ	N_r^*	N_θ^*	u_r^*	w^*
0.4687	1.064	-1.977	-.000	2.536
0.5104	.824	-1.777	.155	2.577
0.5520	.634	-1.628	.298	2.617
0.5937	.479	-1.517	.430	2.657
0.6354	.351	-1.435	.553	2.695
0.6770	.243	-1.374	.667	2.734
0.7187	.151	-1.331	.775	2.774
0.7604	.0707	-1.301	.877	2.814
0.8020	-0.	-1.281	.972	2.856



Case 6.5.2.15 Uniform Pressure
Free $\gamma_1 = 0.999$
Restrained $\gamma_2 = 0.3854$

γ	N_r^*	N_θ^*	U_r^*	w^*
0.3854	3.071	-3.745	-.000	5.002
0.4374	2.305	-3.026	.430	5.042
0.4895	1.765	-2.537	.811	5.072
0.5416	1.369	-2.196	1.152	5.-93
0.5937	1.067	-1.952	1.463	5.108
0.6458	.831	-1.777	1.748	5.119
0.6979	.642	-1.651	2.010	5.128
0.7499	.486	-1.561	2.252	5.137
0.8020	.355	-1.498	2.477	5.146
0.8541	.243	-1.456	2.686	5.156
0.9062	.146	-1.430	2.881	5.169
0.9583	.0615	01.417	3.063	5.185
0.999	.000	-1.414	3.200	5.200



Case 6.5.2.16 Uniform Pressure
Free $\gamma_1 = 1.000$
Restrained $\gamma_2 = 0.4687$

γ	N_r^*	N_θ^*	U_r^*	W^*
0.4687	1.960	-2.712	-0.000	3.644
0.5312	1.439	-2.255	0.353	3.707
0.5937	1.067	-1.952	.664	3.763
0.6562	.790	-1.748	.943	3.813
0.7187	.576	-1.611	1.196	3.861
0.7812	.405	-1.520	1.426	3.909
0.8437	.264	-1.463	1.637	3.958
0.9062	.146	-1.430	1.831	4.010
0.9687	.0456	-1.415	2.010	4.065
1.000	-0.	-1.414	2.094	4.094

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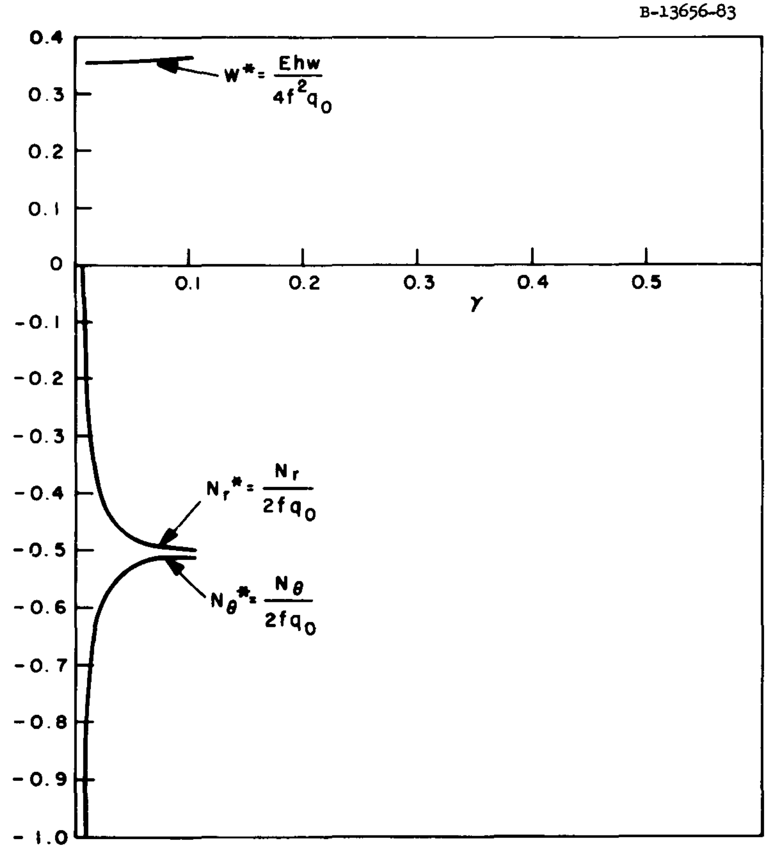

Case 6.5.2.17 Uniform Pressure
Free γ_1 = 0.0104
Restrained γ_2 = 0.1041

γ	N_r^*	N_θ^*	u_r^*	w^*
0.0104	-0.	-1.000	-.0067	.354
0 0208	-.3750	-.625	-.0032	.355
0.0312	-.444	-.556	-.0021	.355
0.0416	-.469	-.532	-.0014	.356
0.0520	-.480	-.522	-.0010	.357
0.0624	-.487	-.516	-.0007	.358
0.0729	-.491	-.514	-.0005	.360
0.0833	-.493	-.512	-.0003	.361
0.0937	-.495	-.512	-.0001	.363
0.1041	-.497	-.513	.000	.365

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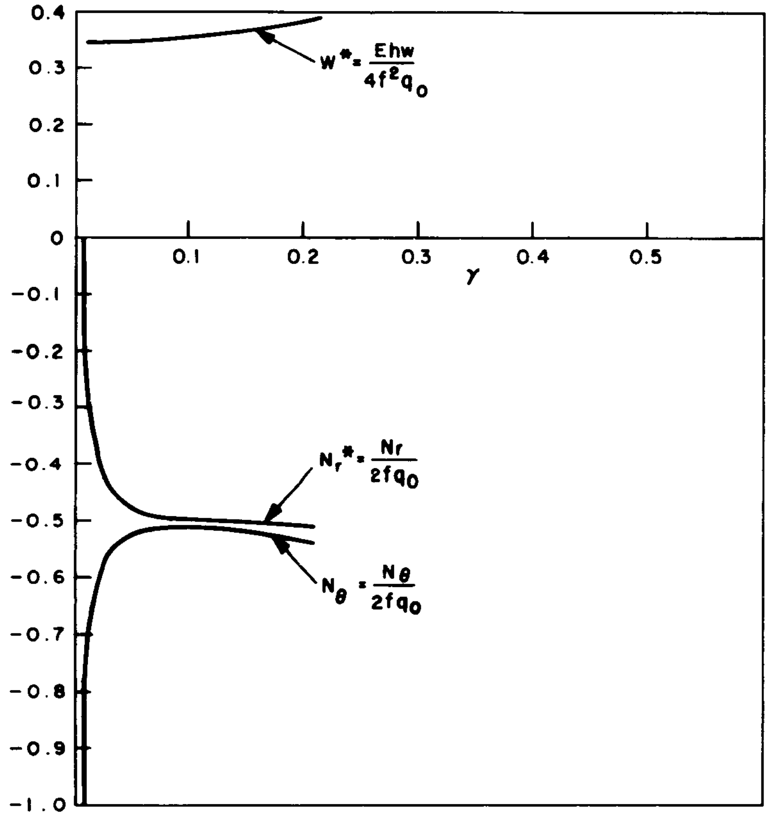

Case 6.5.2.18 Uniform Pressure
Free γ_1 = 0.0104
Restrained γ_2 = 0.2083

γ	N_r^*	N_θ^*	U_r^*	w^*
0.0104	-0.	-1.000	-.0068	.345
0.0312	-.444	-.556	-.0024	.346
0.0520	-.480	-.522	-.0015	.347
0.0729	-.491	-.514	-.0012	.350
0.0937	-.495	-.512	-.0010	.354
0.1145	-.499	-.513	-.0009	.358
0.1354	-.501	-.516	-.0007	.3636
0.1562	-.503	-.520	-.0006	.369
0.1770	-.506	-.524	0.0004	.376
0.1979	-.508	-.530	-.0001	.384
0.2083	-.509	-.533	-.000	.388

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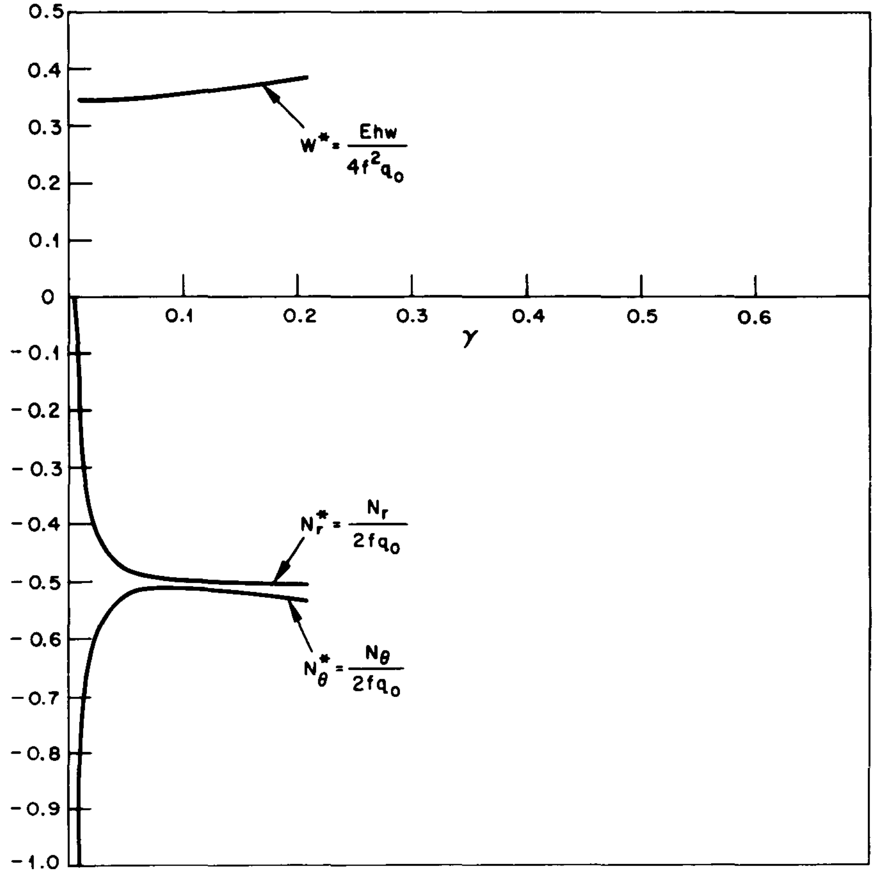
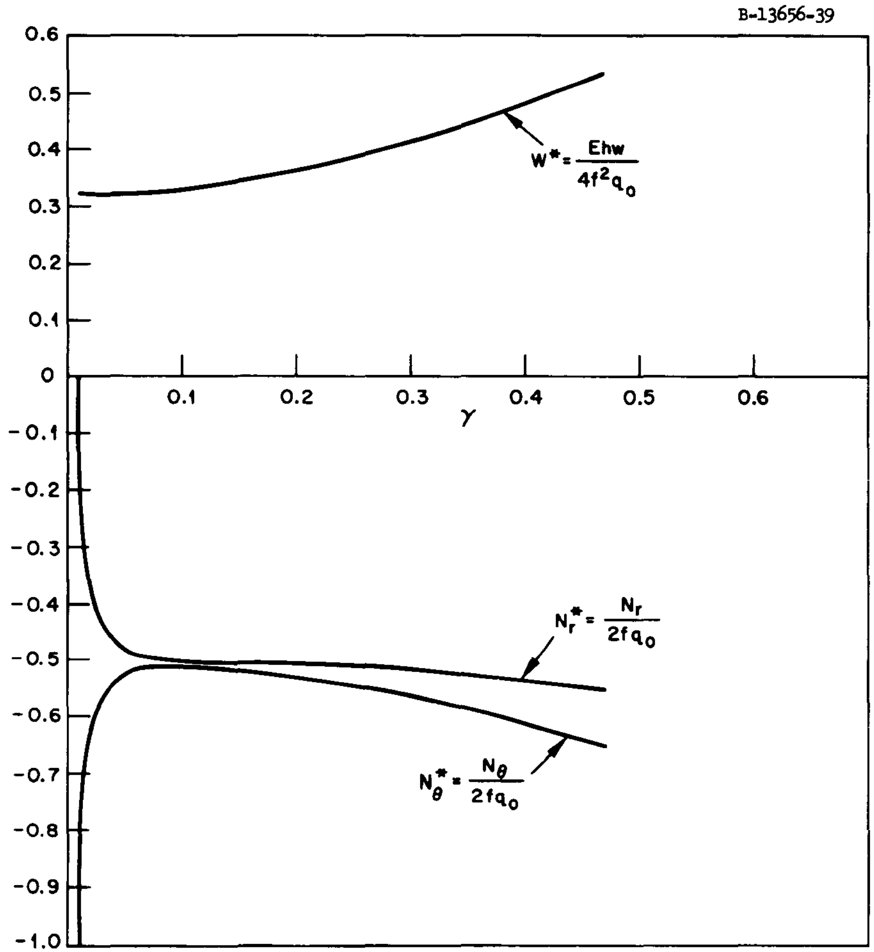


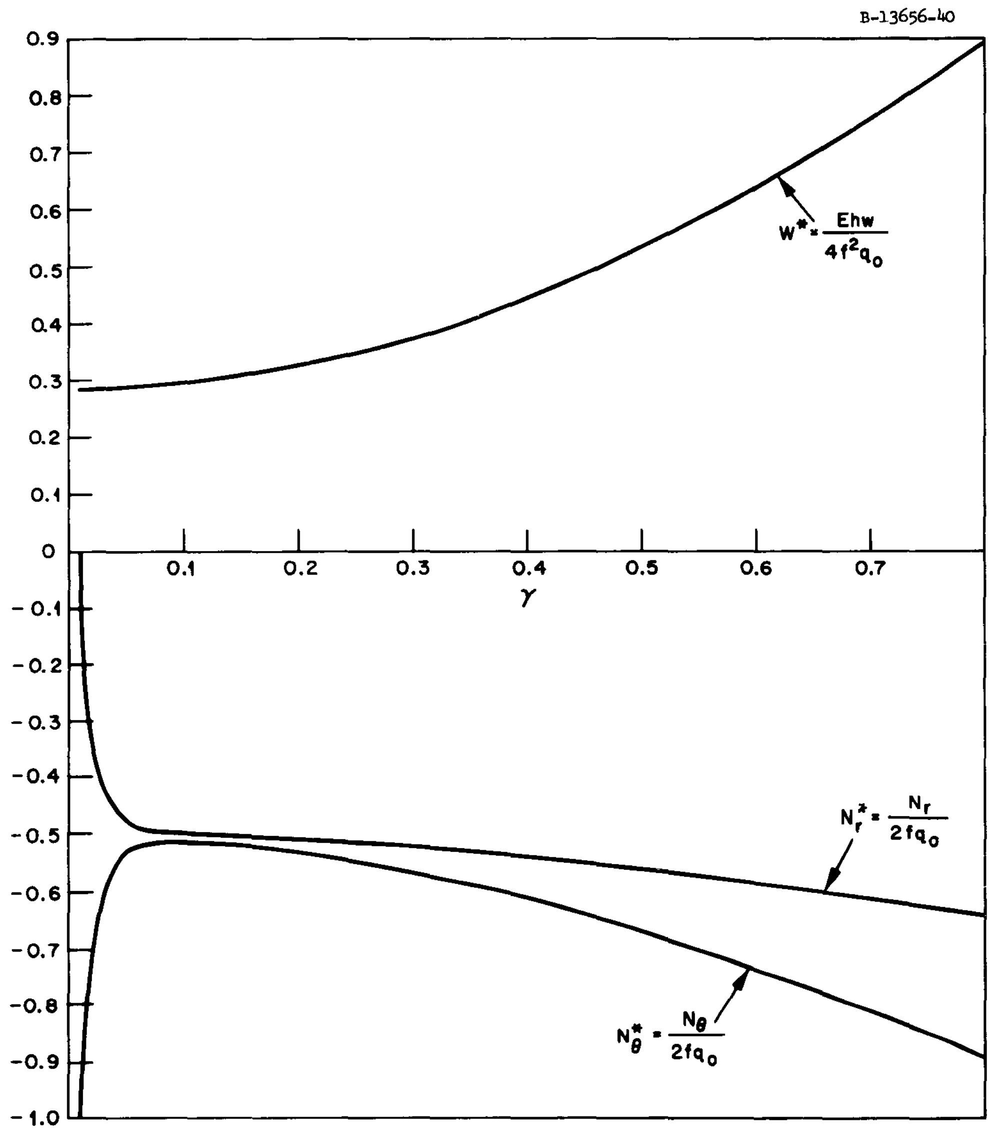
Case 6.5.2.19 Uniform Pressure
Free γ_1 = 0.0104
Restrained γ_2 = 0.2083

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0104	-0.	-1.000	-.0068	0.345
0.0312	-.444	-.556	-.0024	0.346
0.0520	-.480	-.522	-.0015	0.347
0.0729	-.491	-.514	-.0012	0.350
0.0937	-.495	-.512	-.0010	0.354
0.1145	-.499	-.513	-.0009	0.358
0.1354	-.501	-.516	-.0007	0.363
0.1562	-.503	-.520	-.0006	0.369
0.1770	-.506	-.524	-.0004	0.376
0.1979	-.508	-.530	-.0001	0.384
0.2083	-.509	-0.533	-.000	0.388



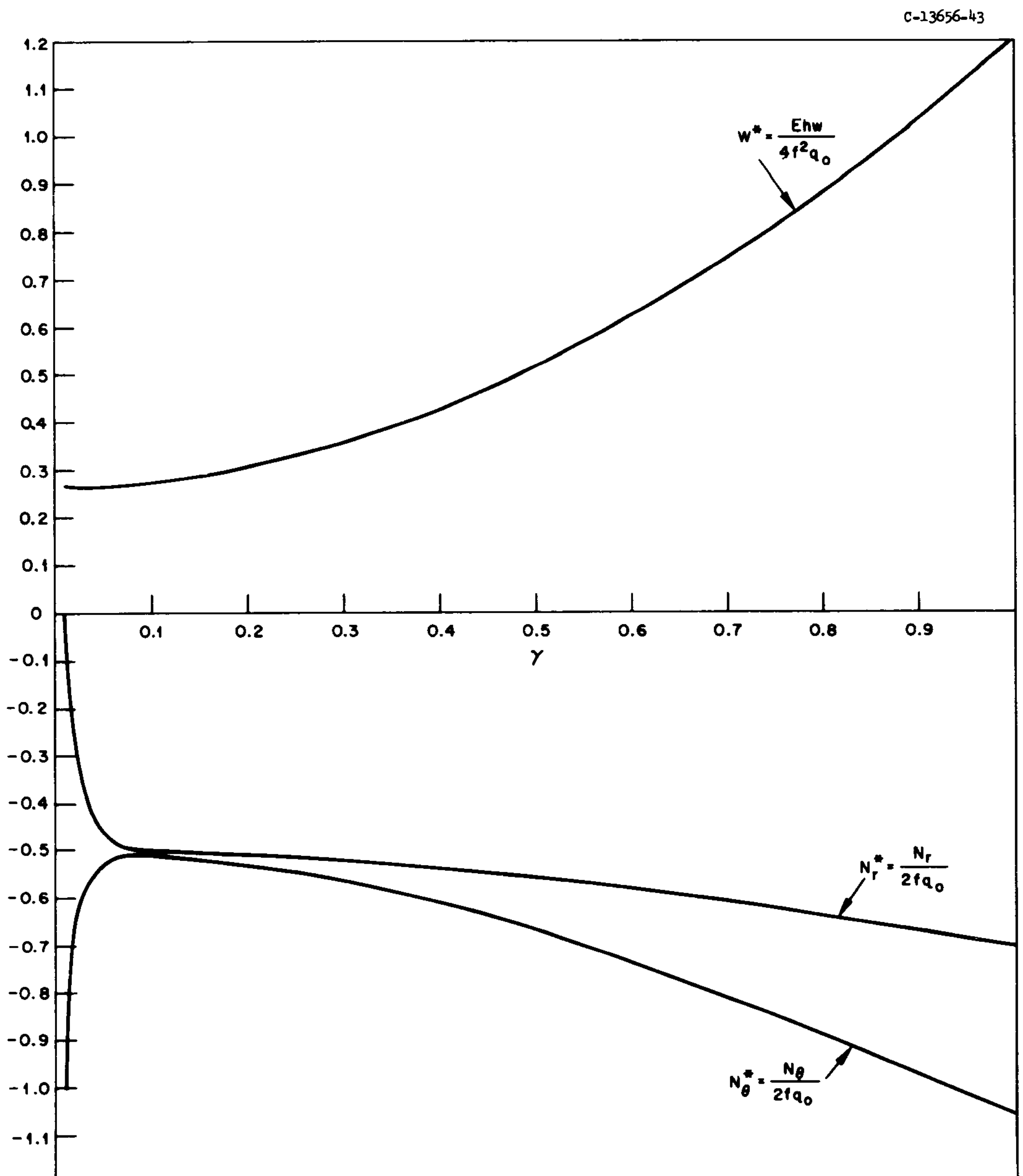
Case 6.5.2.20 Uniform Pressure
Free γ_1 = 0.0104
Restrained γ_2 = 0.4687

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0104	-0.	-1.000	-.0070	.321
0.0758	-.491	-.513	-.0030	.327
0.1413	-.502	-.517	-.0041	.341
0.2068	-.509	-.532	-.0049	.364
0.2723	-.517	-.554	-.0050	.395
0.3377	-.527	-.582	-.0042	.434
0.4032	-.538	-.614	-.0026	.482
0.4687	-.551	-.651	.000	.537



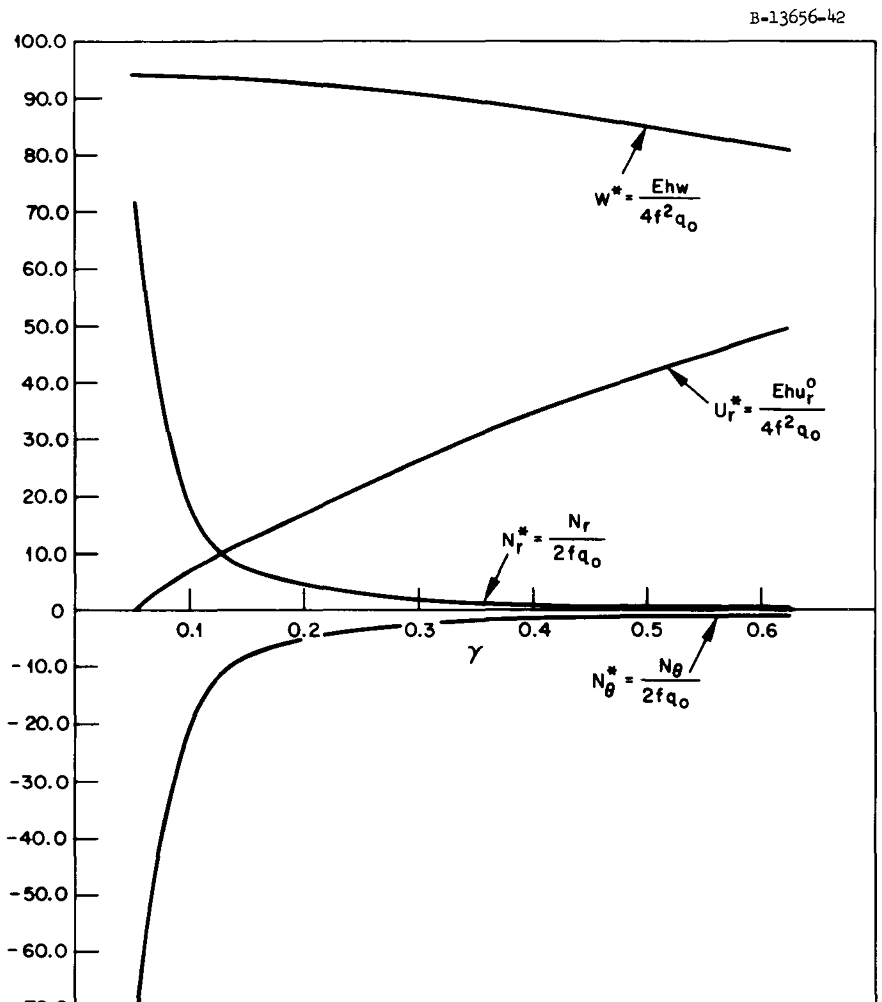
Case 6.5.2.21 Uniform Pressure
Free γ_1 = 0.0104
Restrained γ_2 = 0.8020

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0104	-0.	-1.000	-.0074	.282
0.0937	-.495	-.512	-.0069	.291
0.1770	-.5060	-.524	-.0113	.315
0.2604	-.515	-.550	-.0147	.352
0.3437	-.528	-.585	-.0166	.402
0.4270	-.543	-.627	-.0168	.466
0.5104	-.561	-.677	-.0153	.541
0.5937	-.581	-.733	-.0122	.629
0.6770	-.603	-.793	-.0079	.728
0.7604	-.628	-.858	-.0027	.837
0.8020	-.640	-.891	.000	.896



Case 6.5.2.22 Uniform Pressure
Free γ_1 - 0.0104
Restrained γ_2 = 0.999

γ	N_r^*	N_θ^*	u_r^*	w^*
0.0104	-0.	-1.000	-.0076	.264
0.1145	-.499	-.513	-.0101	.278
0.2187	-.510	-.536	-.0170	.314
0.3229	-.524	-.575	-.0218	.371
0.4270	-.543	-.627	-.0239	.449
0.5312	-.565	-.690	-.0231	.546
0.6354	-.592	-.762	-.0199	.662
0.7395	-.621	-.841	-.0148	.794
0.8437	-.654	-.926	-.0088	.944
0.9479	-.688	-1.015	-.0027	1.110
0.9999	-.707	-1.060	.000	1.200



Case 6.5.2.23 Uniform Pressure
Free γ_1 = 0.0520
Restrained γ_2 = 0.625

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0520	71.596	-72.404	-.000	94.010
0.1562	7.591	-8.422	12.887	93.312
0.2604	2.459	-3.336	22.751	91.574
0.3645	1.031	-1.975	31.584	89.063
0.4687	.429	-1.456	39.491	85.998
0.5729	.109	-1.234	46.477	82.586
0.625	.000	-1.179	49.634	80.806

6.5.3 SHELL CLOSED AT THE APEX

If the shell is closed at the apex, i.e., the shell "begins" at $\gamma = 0$, then in order for the force resultants to remain finite at $\gamma = 0$ (see equation 6.5.4) , we must have

$$C_1 = 0. \tag{6.5.3.1}$$

Then, if the edge, γ_2 , is restrained against tangential displacement, the constant C_2 is given by the expression

$$C_2 = -\frac{\sqrt{1+(\gamma_2)^2}}{3}\left[(2\nu-1)(\gamma_2)^2 + 2(\nu+1)\right]. \tag{6.5.3.2}$$

Six different sizes of shells have been analyzed. It is interesting to note that the presence of a moderate size hole at the apex of a shell has very little influence on the normal displacements even though the distributions of the force resultants are appreciably different (compare, e.g., cases 6.5.1.1, 6.5.2.5, 6.5.3.1).

A list of the cases which have been analyzed is contained in Table 6.5.3.1

Table 6.5.3.1

Case	Restrained at γ_2
6.5.3.1	.6250
6.5.3.2	.3854
6.5.3.3	.5357
6.5.3.4	.2678
6.5.3.5	.8035
6.5.3.6	1.0267

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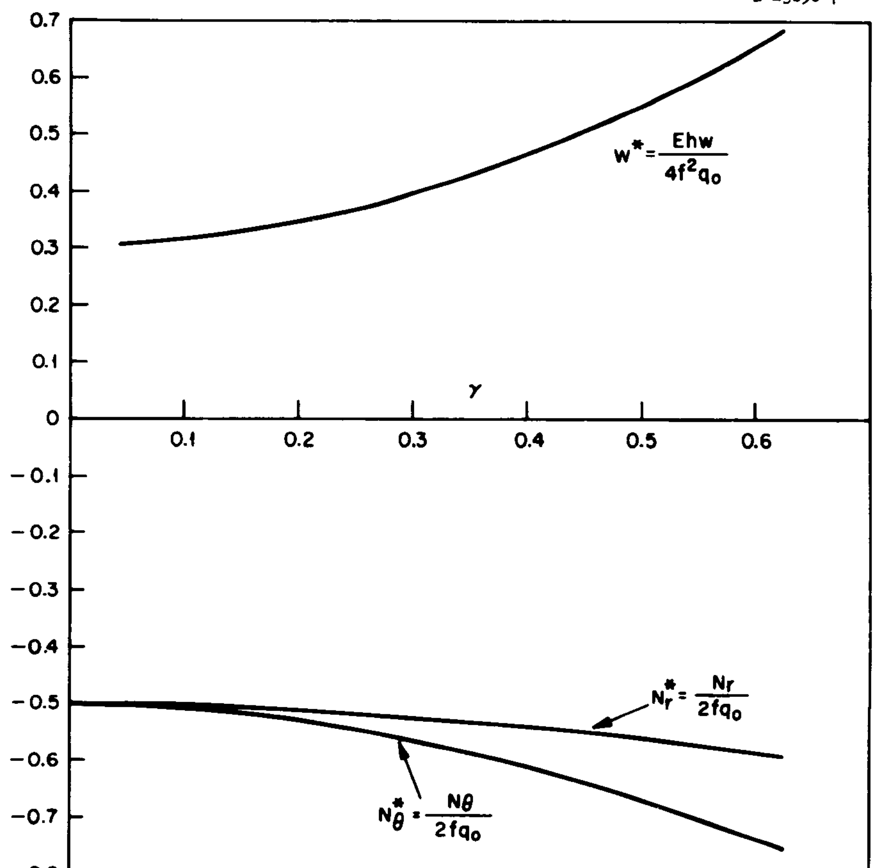

CASE 6.5.3.1 UNIFORM PRESSURE
CLOSED AT APEX.
RESTRAINED γ = .625

γ	N_r	N_θ^*	U_r^*	w^*
0	− .500	−.500	0	
0.0446			−.002	.305
0.1306	− .504	−.513	−.0058	.320
0.2166	− .512	−.534	−.0085	.351
0.3025	− .522	−.566	−.0098	.395
0.3885	− .536	−.607	−.0095	.454
0.4745	− .553	−.655	−.0074	.526
0.5605	− .573	−.710	−.0037	.610
0.6249	− .590	−.755	.000	.682

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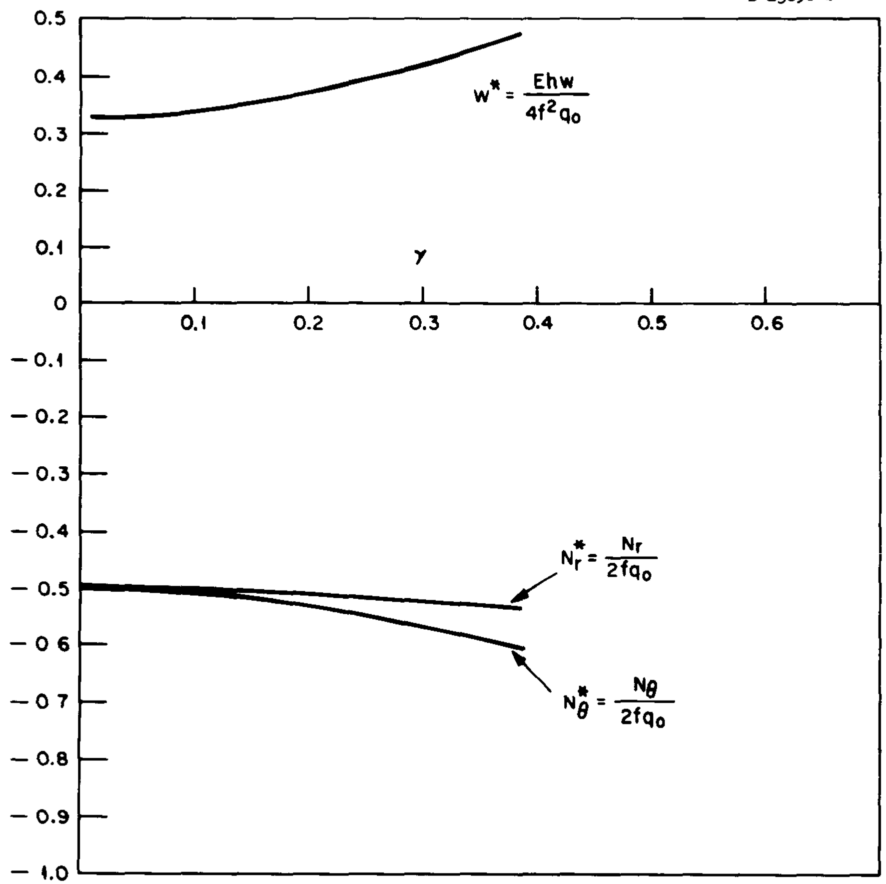

CASE 6.5.3.2 UNIFORM PRESSURE
CLOSED AT APEX.
RESTRAINED $\gamma = .3854$

γ	N_r^*	N_θ^*	U_r^*	w^*
0	− .500	− .500	0	
0.0104			−.000213	.330
0.0520	− .501	− .502	−.00104	.332
0.0937	− .502	− .506	−.00179	.338
0.1354	− .504	− .514	−.00238	.348
0.1770	− .508	− .523	−.00276	.361
0.2187	− .512	− .535	−.00288	.378
0.2604	− .517	−.549	−.00269	.397
0.3020	− .522	−.566	−.00216	.420
0.3437	− .529	−.584	−.00127	.447
0.3854	− .536	−.605	.000	.476

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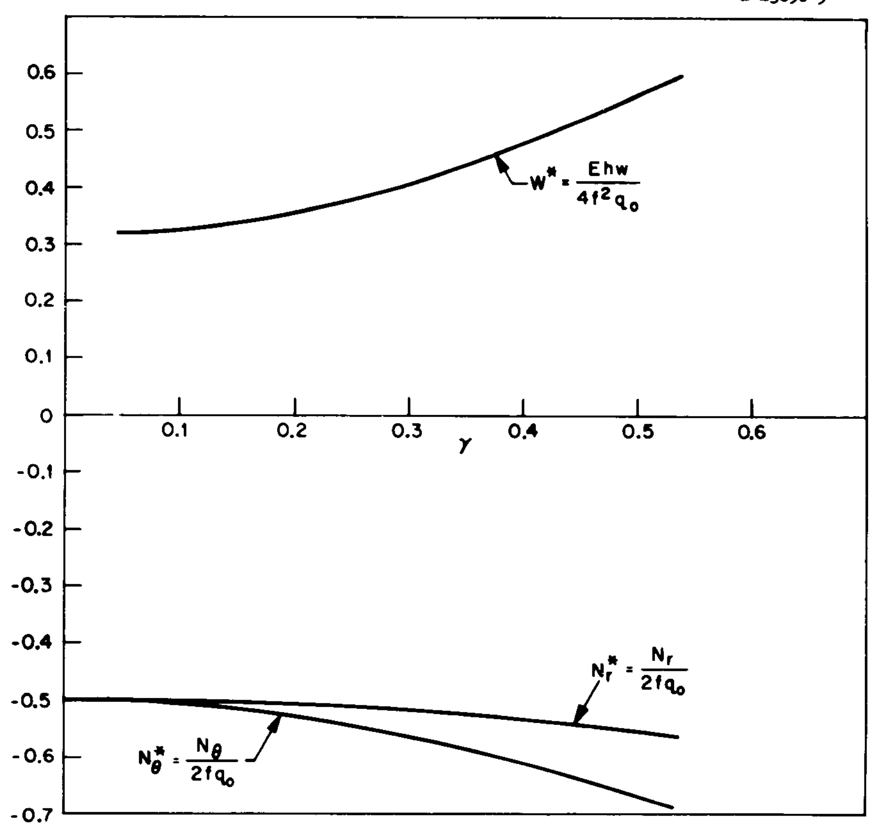

CASE 6.5.3.3 UNIFORM PRESSURE
CLOSED AT APEX
RESTRAINED $\gamma_2 = 0.5357$

γ	N_r^*	N_θ^*	U_r^*	W^*
0	-0.500	-0.500	0	
0.0669	-0.501	-0.503	-0.002	-0.318
0.1116	-0.503	-0.509	-0.003	-0.326
0.1562	-0.506	-0.518	-0.005	-0.338
0.2008	-0.509	-0.529	-0.0060	-0.354
0.2455	-0.514	-0.544	-0.0066	-0.374
0.2901	-0.520	-0.561	-0.0068	-0.398
0.3348	-0.527	-0.580	-0.0066	-0.425
0.3794	-0.534	-0.602	-0.0059	-0.456
0.4241	-0.543	-0.625	-0.004	-0.491
0.4687	-0.552	-0.651	-0.003	-0.529
0.513	-0.562	-0.679	-0.001	-0.571

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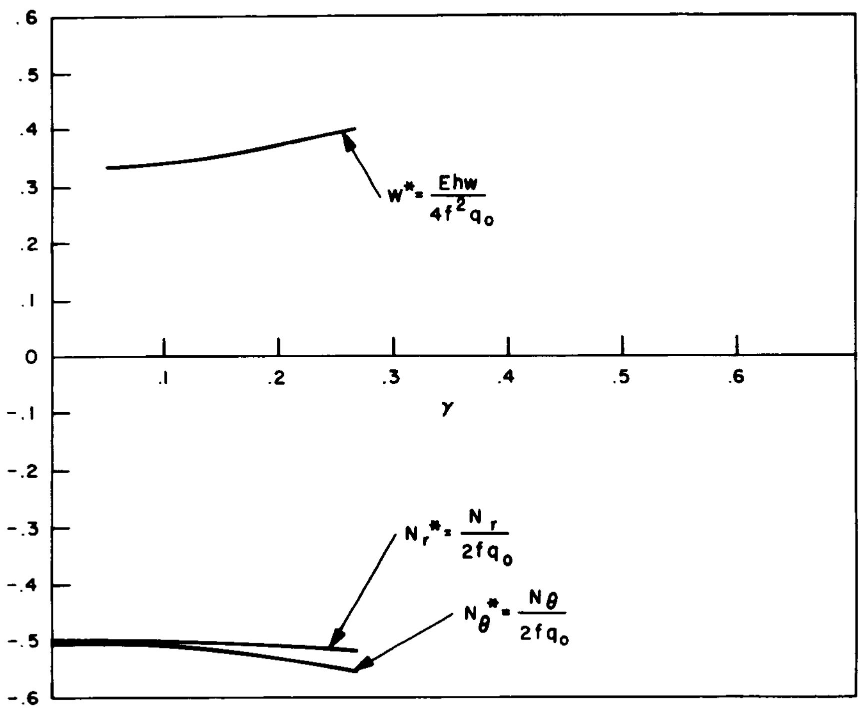
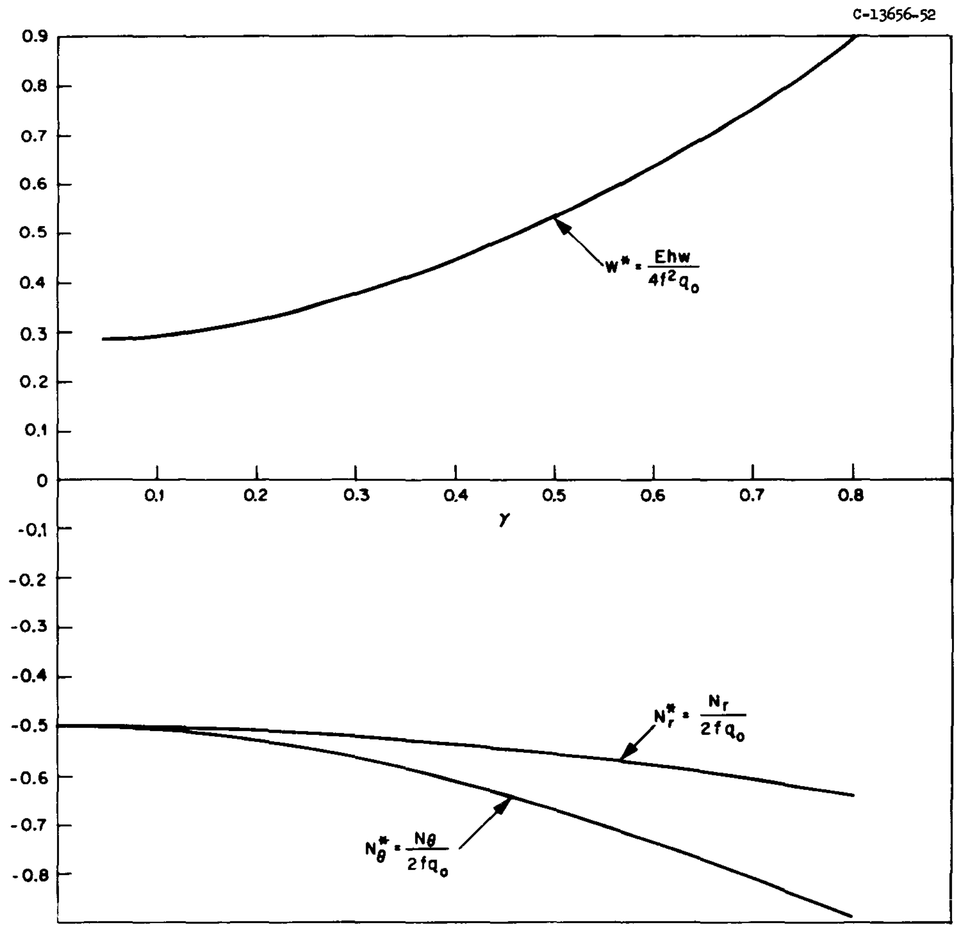


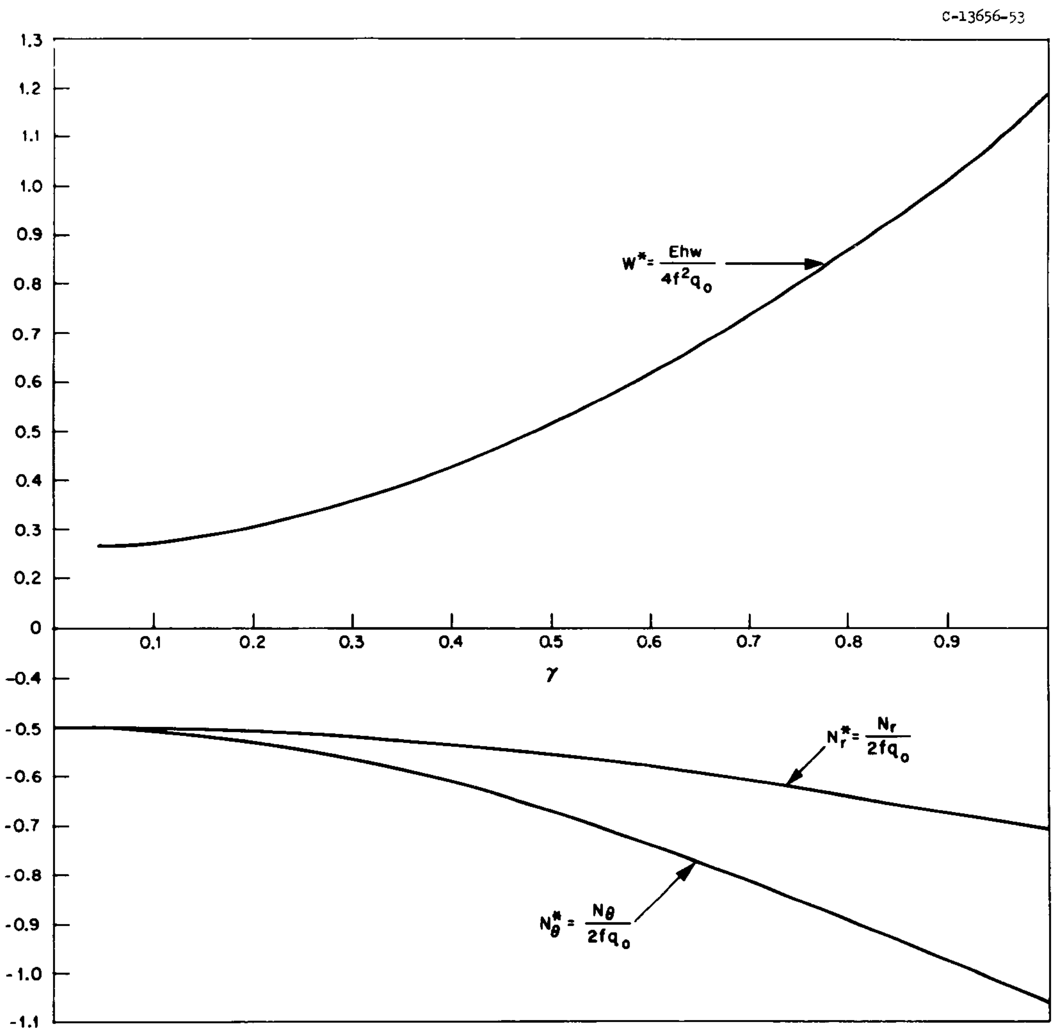
CASE 6.5.3.4 UNIFORM PRESSURE
CLOSED AT APEX
RESTRAINED $\gamma = .2678$

γ	N_r^*	N_θ^*	u^*	w^*
0	-.500	-.500	0	—
.0446	-.500	-.501	-.0004	.341
.0669	-.501	-.503	-.0006	.344
.0892	-.501	-.505	-.0008	.347
.1116	-.503	-.509	-.0009	.352
.1339	-.504	-.513	-.00101	.357
.1562	-.506	-.518	-.00103	.364
.1785	-.507	-.523	-.00098	.371
.2008	-.509	-.529	-.0008	.379
.2232	-.512	-.536	-.00066	.389
.2455	-.514	-.544	-.00037	.399
.2678	-.517	-.552	-.000	.410



Case 6.5.3.5 Uniform Pressure
Closed at Apex
Restrained $\gamma_2 = 0.8035$

γ	N_r^*	N_θ^*	U_r^*	w^*
0.	-0.5	-0.5	0.	--
0.0892	-.501	-.505	-.0058	.290
0.1785	-.507	-.523	-.0110	.315
0.2678	-.517	-.552	-.0147	.355
0.3571	-.530	-.590	-.0166	.411
0.4464	-.547	-.638	-.0165	.482
0.5357	-.567	-.693	-.0145	.566
0.6249	-.589	-.755	-.0108	.664
0.7142	-.614	-.822	-.0057	.775
0.8035	-.641	-.893	.000	.898



Case 6.5.3.6 Uniform Pressure
Closed at Apex
Restrained γ_2 = 1.0267

γ	N_r^*	N_θ^*	U_r^*	W^*
0.0	-0.5	-0.5	0.	--
0.8092	-.501	-.505	-.0076	-.271
0.1785	-.507	-.523	-.0144	.296
0.2678	-.517	-.552	-.0198	.336
0.3571	-.530	-.590	-.0232	.393
0.4464	-.547	-.638	-.0245	.464
0.5357	-.567	-.693	-.0238	.549
0.6249	-.589	-.755	-.0212	.648
0.7142	-.614	-.822	-.0172	.759
0.8035	0.641	-.893	-.0123	.883
0.8928	-.670	-.967	-.0071	1.019
0.9821	-.700	-1.044	-.0021	1.167
1.0267	-.716	-1.084	.000	1.246

6.6 THE ASYMMETRIC MEMBRANE BEHAVIOR: GRAVITY LOADS

We now turn to the more difficult task of calculating the behavior of the paraboloidal shell of revolution under its own dead weight in the more general case when the axis of the paraboloid does not coincide with the axis of gravity (see figure 6.4.1). This means we must deal with the full set of membrane equations, 6.1.1 through 6.1.9, in which the loads are prescribed by equations 6.3.5, 6.3.6, and 6.3.7.

As a result of the closed character of the shell in the circumferential direction, the surface loading as well as the edge loading must be periodic functions of the angle θ with a period of 2π (see equations 6.3.5, 6.3.6, and 6.3.7). Therefore we have as eigenfunctions the set of trigonometric functions $\cos n\theta$ and $\sin n\theta$ for n=0, 1, 2, ----. It is well known, in the elementary theory of Fourier Series, that this set of eigenfunctions is complete. As N_r, N_θ, and $N_{r\theta}$ are sufficiently smooth, the usual expansion theorem applies.[20, 21] Thus we can write

$$N_r(\gamma, \theta) = a_0(\gamma) + \sum_{n=1}^{\infty} \left[a_n(\gamma) \sin n\theta + d_n(\gamma) \cos n\theta \right], \quad 6.6.1$$

$$N_\theta(\gamma, \theta) = b_0(\gamma) + \sum_{n=1}^{\infty} \left[b_n(\gamma) \sin n\theta + e_n(\gamma) \cos n\theta \right], \quad 6.6.2$$

$$N_{r\theta}(\gamma, \theta) = f_0(\gamma) + \sum_{n=1}^{\infty} \left[f_n(\gamma) \sin n\theta + c_n(\gamma) \cos n\theta \right]. \quad 6.6.3$$

Substitution of these expressions into equations 6.1.4 through 6.1.6 and with the loads given by equations 6.3.5, 6.3.6, 6.3.7 yields the following equations (primes indicate differentiation with respect to γ):

$$\gamma a_0' + a_0 - b_0 = 2 f \rho_0 h \cos \psi \left\{ \gamma^2 \right\}, \quad 6.6.4$$

$$\frac{a_0}{1 + (\gamma)^2} + b_0 = 2 f \rho_0 h \cos \psi, \quad 6.6.5$$

$$\gamma f_0' + 2 f_0 = 0, \quad 6.6.6$$

$$\gamma a_1' + a_1 - c_1\sqrt{1+(\gamma)^2} - b_1 = 2f\rho_0 h \sin\psi\{\gamma\}, \qquad 6.6.7$$

$$\gamma d_1' + d_1 + \sqrt{1+(\gamma)^2}\ f_1 - e_1 = 0, \qquad 6.6.8$$

$$\gamma f_1' + f_1 - e_1\sqrt{1+(\gamma)^2} = 0, \qquad 6.6.9$$

$$\gamma C_1' + 2c_1 + b_1\sqrt{1+(\gamma)^2} = -2f\rho_0 h \sin\psi\left\{\gamma\sqrt{1+(\gamma)^2}\right\}, \qquad 6.6.10$$

$$\frac{a_1}{1+(\gamma)^2} + b_1 = 2f\rho_0 h \sin\psi\{\gamma\}, \qquad 6.6.11$$

$$\frac{d_1}{1+(\gamma)^2} + e_1 = 0, \qquad 6.6.12$$

$$\gamma a_n' + a_n - n\,c_n\sqrt{1+(\gamma)^2} - b_n = 0, \qquad 6.6.13$$

$$\gamma d_n' + d_n + n\sqrt{1+(\gamma)^2}\; f_n - e_n = 0, \qquad 6.6.14$$

$$\gamma f_n' + 2f_n - n e_n \sqrt{1+(\gamma)^2} = 0, \qquad 6.6.15$$

$$\gamma C_n' + 2C_n + n b_n \sqrt{1+(\gamma)^2} = 0, \qquad 6.6.16$$

$$\frac{a_n}{1+(\gamma)^2} + b_n = 0, \qquad 6.6.17$$

and

$$\frac{d_n}{1+(\gamma)^2} + d_n = 0\,. \qquad 6.6.18$$

where n=2,3,4,---

With the usual boundary conditions imposed upon the stress resultants, we show next that equations 6.6.1, 6.6.2, and 6.6.3 must reduce to

$$N_r(\gamma,\theta) = a_0(\gamma) + a_1(\gamma)\sin\theta, \qquad 6.6.19$$

$$N_\theta(\gamma,\theta) = b_0(\gamma) + b_1(\gamma)\sin\theta, \qquad 6.6.20$$

$$N_{r\theta}(\gamma,\theta) = C_1(\gamma)\cos\theta. \qquad 6.6.21$$

For a shell with a free edge at $\gamma = \gamma_1$, the homogeneous differential equation 6.6.6 for f_0 implies

$$\left[(\gamma)^2 f_0\right]' = 0 \qquad 6.6.22$$

or

$$f_0(\gamma) = \frac{A}{(\gamma)^2} \qquad 6.6.23$$

where A is a constant of integration. But the free edge condition (see equation 6.2.6) requires the shear resultant $N_{r\theta}$ to be zero at γ_1 . We must have (see equation 6.6.3)

$$A = 0. \qquad 6.6.24$$

Therefore, it follows that

$$f_0(\gamma) \equiv 0. \qquad 6.6.25$$

Next, equations 6.6.8, 6.6.9, and 6.6.12 can be uncoupled to yield the following relations:

$$f_1(\gamma) = \frac{-1}{\sqrt{1+(\gamma)^2}} \left\{ \gamma d_1' + \left(1+\frac{1}{1+(\gamma)^2}\right) d_1 \right\}, \qquad 6.6.26$$

$$e_1(\gamma) = \frac{-d_1}{1+(\gamma)^2}, \qquad 6.6.27$$

$$d_1'' + \left(\frac{5}{\gamma} - \frac{2\gamma}{1+(\gamma)^2}\right) d_1' + \left(\frac{3}{(\gamma)^2\,[1+(\gamma)^2]^2}\right) d_1 = 0. \qquad 6.6.28$$

Let the most general solution of the homogeneous equation 6.6.28 be

$$d_1(\gamma) = g_1 h_1(\gamma) + g_2 h_2(\gamma) \qquad 6.6.29$$

where g_1 and g_2 are non-zero real constants and $h_1(\gamma)$ and $h_2(\gamma)$ are linearly independent solutions of equation 6.6.28.

From equation 6.6.26

$$f_1(\gamma) = \frac{-1}{\sqrt{1+(\gamma)^2}} \left\{ g_1\left[\gamma h_1'(\gamma) + \left(1 + \frac{1}{1+(\gamma)^2}\right)h_1(\gamma)\right] + g_2\left[\gamma h_2'(\gamma) + \left(1+\frac{1}{1+(\gamma)^2}\right)h_2(\gamma)\right]\right\}. \qquad 6.6.30$$

From the boundary conditions, equations 6.2.5 and 6.2.6, we require

$$g_1 h_1(\gamma_1) + g_2 h_2(\gamma_1) = 0 \qquad 6.6.31$$

and

$$g_1\left[\gamma_1 h_1'(\gamma_1) + \left(1+\frac{1}{1+(\gamma_1)^2}\right)h_1(\gamma_1)\right] + g_2\left[\gamma_1 h_2'(\gamma_1) + \left(1+\frac{1}{1+(\gamma_1)^2}\right)h_2(\gamma_1)\right] = 0. \qquad 6.6.32$$

These latter two equations which are simultaneous homogeneous equations for the constants g_1 and g_2 possess non-trivial solutions only if the determinant of the coefficients of g_1 and g_2 vanishes.

If the determinant of the coefficients of g_1 and g_2 does vanish, we have

$$h_1(\gamma_1)\left[\gamma_1 h_2'(\gamma_1) + \left(1+\frac{1}{1+(\gamma_1)^2}\right) h_2(\gamma_1)\right] - h_2(\gamma_1)\left[\gamma_1 h_1'(\gamma_1) + \left(1+\frac{1}{1+(\gamma_1)^2}\right) h_1(\gamma_1)\right] = 0 \qquad 6.6.33$$

or

$$h_1(\gamma_1)\, h_2'(\gamma_1) - h_2(\gamma_1)\, h_1'(\gamma_1) = 0. \qquad 6.6.34$$

The left hand side of 6.6.34 is the Wronskian of the functions $h_1(\gamma)$ and $h_2(\gamma)$. The vanishing of the Wronskian at $\gamma = \gamma_1$ implies the linear dependence of $h_1(\gamma)$ and $h_2(\gamma)$ for all positive γ and we have reached a contradiction.[18]

Thus we can only have

$$g_1 = g_2 = 0 \qquad 6.6.35$$

or

$$d_1(\gamma) \equiv 0. \qquad 6.6.36$$

Consequently (cf. equation 6.6.26 and 6.6.27)

$$f_1(\gamma) \equiv 0, \qquad 6.6.37$$

$$e_1(\gamma) \equiv 0. \qquad 6.6.38$$

By repetitions of the above argument, we can also show

that

$$d_n(\gamma) \equiv 0, \tag{6.6.39}$$

$$f_n(\gamma) \equiv 0, \tag{6.6.40}$$

and

$$e_n(\gamma) \equiv 0 \tag{6.6.41}$$

for all n and

$$a_n(\gamma) \equiv 0, \tag{6.6.42}$$

$$b_n(\gamma) \equiv 0, \tag{6.6.43}$$

and

$$c_n(\gamma) \equiv 0 \tag{6.6.44}$$

for $n \geq 2$.

In short we have the very desirable results that

$$N_r(\gamma, \theta) = a_0(\gamma) + a_1(\gamma)\sin\theta, \tag{6.6.45}$$

$$N_\theta(\gamma, \theta) = b_0(\gamma) + b_1(\gamma)\sin\theta, \tag{6.6.46}$$

and

$$N_{r\theta} = c_1(\gamma)\cos\theta. \tag{6.6.47}$$

If the shell is closed at the apex, the usual condition imposed upon the stress resultants is that they be finite.[10,11] The

application of this condition to equation 6.6.23 results in

$$f_0(\gamma) \equiv 0. \tag{6.6.48}$$

Next we observe that equation 6.6.28 has a regular singularity at $\gamma = 0$.

Therefore, in the neighborhood of the apex, the most general solution of 6.6.28 takes the form[18, 19]

$$d_1(\gamma) = C_1 \sum_{n=0}^{\infty} s_n \gamma^{n+m_1} + C_2 \left\{ \ln(\gamma) \sum_{n=0}^{\infty} s_n \gamma^{n+m_1} + \sum_{n=0}^{\infty} t_n \gamma^{n+m_2} \right\} \tag{6.6.49}$$

where C_1 and C_2 are constants of integration to be determined by the known side constraints, s_n and t_n are constants fixed by the relevant recurrence relations, m_1 and m_2 ($m_1 \geq m_2$) are roots of the indicial equation

$$m^2 + 4m + 3 = 0, \tag{6.6.50}$$

or

$$m_1 = -1, \tag{6.6.51}$$

and

$$m_2 = -3. \tag{6.6.52}$$

To ensure finiteness at the apex, we must have

$$C_1 = C_2 = 0 \tag{6.6.53}$$

that

$$d_n(\gamma) \equiv 0, \qquad 6.6.39$$

$$f_n(\gamma) \equiv 0, \qquad 6.6.40$$

and

$$e_n(\gamma) \equiv 0 \qquad 6.6.41$$

for all n and

$$a_n(\gamma) \equiv 0, \qquad 6.6.42$$

$$b_n(\gamma) \equiv 0, \qquad 6.6.43$$

and

$$c_n(\gamma) \equiv 0 \qquad 6.6.44$$

for $n \geq 2$.

In short we have the very desirable results that

$$N_r(\gamma, \theta) = a_0(\gamma) + a_1(\gamma) \sin \theta, \qquad 6.6.45$$

$$N_\theta(\gamma, \theta) = b_0(\gamma) + b_1(\gamma) \sin \theta, \qquad 6.6.46$$

and

$$N_{r\theta} = c_1(\gamma) \cos \theta. \qquad 6.6.47$$

If the shell is closed at the apex, the usual condition imposed upon the stress resultants is that they be finite.[10,11] The

and consequently

$$d_1(\gamma) \equiv 0, \qquad 6.6.54$$

$$e_1(\gamma) \equiv 0, \qquad 6.6.55$$

and

$$f_1(\gamma) \equiv 0. \qquad 6.6.56$$

Repetitions of the above argument will lead to the conclusion that $a_n(\gamma)$, $b_n(\gamma)$, $c_n(\gamma)$, $d_n(\gamma)$, $e_n(\gamma)$, and $f_n(\gamma)$ must also vanish identically for $n \geq 2$. We have again arrived at the very desirable results that

$$N_r(\gamma,\theta) = a_0(\gamma) + a_1(\gamma) \sin\theta, \qquad 6.6.57$$

$$N_\theta(\gamma,\theta) = b_0(\gamma) + b_1(\gamma) \sin\theta, \qquad 6.6.58$$

and

$$N_{r\theta}(\gamma,\theta) = c_1(\gamma) \cos\theta. \qquad 6.6.59$$

Thus the problem of the membrane stress resultants is reduced to the ascertainment of solutions to the system of differential

equations (6.6.4), (6.6.5), (6.6.7), (6.6.10) and (6.6.11). Equations 6.6.4 and 6.6.5 completely determine $a_0(\gamma)$ and $b_0(\gamma)$ up to a constant of integration. It should be observed that these are the portions of the N_r and N_ϕ induced by the axi-symmetric component of the load. This axi-symmetric portion has the same dependence on γ except for a factor $\cos\psi$ as was found earlier for the completely axi-symmetric behavior (cf. equations 6.4.14 and 6.4.15).

Thus

$$a_0(\gamma) = \frac{2f\rho_0 h \cos\psi}{3} \frac{\sqrt{1+(\gamma)^2}}{(\gamma)^2} \left\{ \left[1+(\gamma)^2\right]^{3/2} + C_1 \right\} \qquad 6.6.60$$

and

$$b_0(\gamma) = 2f\rho_0 h \cos\psi \left\{ 1 - \frac{1}{3(\gamma)^2\sqrt{1+(\gamma)^2}} \left[\left[1+(\gamma)^2\right]^{3/2} + C_1\right] \right\} \qquad 6.6.61$$

where C_1 is a constant of integration to be determined by the side condition of the shell.

The remaining three equations, 6.6.7, 6.6.10, 6.6.11, can be reduced to the following three uncoupled equations

$$a_1'' + \left(\frac{5}{\gamma} - \frac{2\gamma}{1+(\gamma)^2}\right) a_1' + \frac{3}{(\gamma)^2[1+(\gamma)^2]^2} \; a_1 = -4f\rho_0 h \sin\psi \left\{\frac{1+(\gamma)^2}{\gamma}\right\},$$

6.6.62

$$b_1(\gamma) = \frac{-a_1(\gamma)}{1+(\gamma)^2} + 2f\rho_0 h \sin\psi \left\{\gamma\right\},$$

6.6.63

$$c_1(\gamma) = \frac{1}{\sqrt{1+(\gamma)^2}} \left\{\gamma\, a_1'(\gamma) + \left(1 + \frac{1}{1+(\gamma)^2}\right) a_1(\gamma)\right\}$$

6.6.64

and we need only to solve the differential equation 6.6.62 to complete the analysis of membrane stresses.

Observe that (6.6.62) can be rewritten as

$$\frac{\sqrt{1+(\gamma)^2}}{(\gamma)^2} \frac{d}{d\gamma}\left\{\frac{1}{\gamma}\frac{d}{d\gamma}\left(\frac{(\gamma)^3 a_1}{\sqrt{1+(\gamma)^2}}\right)\right\} = -4f\rho_0 h \sin\psi \left\{\frac{1+(\gamma)^2}{\gamma}\right\}$$

or

$$\frac{d}{d\gamma}\left\{\frac{1}{\gamma}\frac{d}{d\gamma}\left(\frac{(\gamma)^3 a_1}{\sqrt{1+(\gamma)^2}}\right)\right\} = -4f\rho_0 h \sin\psi \left\{\gamma\sqrt{1+(\gamma)^2}\right\}.$$

6.6.65

Thus

$$\frac{d}{d\gamma}\left(\frac{(\gamma)^3 a_1}{\sqrt{1+(\gamma)^2}}\right) = -4f\rho_0 h \sin\psi \left\{\frac{\left(1+(\gamma)^2\right)^{3/2}}{3} + C_3\right\}\left\{\gamma\right\},$$

6.6.66

$$\frac{(\gamma)^3 a_1}{\sqrt{1+(\gamma)^2}} = -4f\rho_0 h \sin\psi \left\{ C_4 + \frac{C_3(\gamma)^2}{2} + \frac{(1+(\gamma)^2)^{5/2}}{15} \right\},$$

6.6.67

and

$$a_1(\gamma) = -4f\rho_0 h \sin\psi \left\{ \frac{C_4\sqrt{1+(\gamma)^2}}{(\gamma)^3} + \frac{C_3\sqrt{1+(\gamma)^2}}{2\gamma} + \frac{(1+(\gamma)^2)^3}{15(\gamma)^3} \right\},$$

6.6.68

where C_3 and C_4 are constants of integration.

$b_1(\gamma)$ and $c_1(\gamma)$ can then be computed with the help of equations (6.6.63) and (6.6.64) to be

$$b_1(\gamma) = 2f\rho_0 h \sin\psi \left\{ \gamma + \frac{2}{(\gamma)^3\sqrt{1+(\gamma)^2}} \left[C_4 + \frac{C_3(\gamma)^2}{2} + \frac{(1+(\gamma)^2)^{5/2}}{15} \right] \right\},$$

6.6.69

$$c_1(\gamma) = 4f\rho_0 h \sin\psi \left\{ \frac{\sqrt{1+(\gamma)^2}}{15(\gamma)^3} \left(1 - 4(\gamma)^2\right) - \frac{C_3}{2\gamma} + \frac{C_4}{(\gamma)^3} \right\}.$$

6.6.70

Finally,, from (6.6.19), (6.6.20), and (6.6.21), we have

$$N_r(\gamma,\theta) = \frac{2f\rho_0 h \cos\psi}{3} \quad \frac{\sqrt{1+(\gamma)^2}}{(\gamma)^2} \left\{ (1+(\gamma)^2)^{3/2} + C_1 \right\}$$
$$-4f\rho_0 h \sin\psi \left\{ \frac{C_3\sqrt{1+(\gamma)^2}}{2\gamma} + \frac{C_4\sqrt{1+(\gamma)^2}}{(\gamma)^3} + \frac{(1+(\gamma)^2)^3}{15(\gamma)^3} \right\} \sin\theta,$$

6.6.71

$$N_\theta(\gamma,\theta) = 2f\rho_0 h\cos\psi\left\{1 - \frac{1}{3(\gamma)^2\sqrt{1+(\gamma)^2}}\left[\left(1+(\gamma)^2\right)^{3/2} + C_1\right]\right\}$$

$$+ 2f\rho_0 h\sin\psi\left\{\gamma + \frac{2}{(\gamma)^3\sqrt{1+(\gamma)^2}}\left[C_4 + \frac{C_3(\gamma)^2}{2} + \frac{\left(1+(\gamma)^2\right)^{5/2}}{15}\right]\right\}\sin\theta, \qquad 6.6.72$$

and

$$N_{r\theta}(\gamma,\theta) = 4f\rho_0 h\sin\psi\left\{\frac{\left(1+(\gamma)^2\right)^{3/2}}{15(\gamma)^3}\left(1-4(\gamma)^2\right) - \frac{C_3}{2\gamma} + \frac{C_4}{(\gamma)^3}\right\}\cos\theta. \qquad 6.6.73$$

To determine the displacements, the stress-strain relations (cf. equations 6.1.7, 6.1.8, 6.1.9) and the strain-displacement relations (cf. equations 6.1.1, 6.1.2, 6.1.3) are combined to yield the following set of partial differential equations for the displacements u_r^o, u_θ^o and w :

$$\frac{\partial u_r^o}{\partial \gamma} - \frac{w}{1+(\gamma)^2} = \frac{2f\sqrt{1+(\gamma)^2}}{Eh}\left[N_r - \nu N_\theta\right] = 2f\sqrt{1+(\gamma)^2}\ \epsilon_r^o, \qquad 6.6.74$$

$$\sqrt{1+(\gamma)^2}\,\frac{\partial u_\theta^o}{\partial\theta} + u_r^o - \gamma w = \frac{2f\gamma\sqrt{1+(\gamma)^2}}{Eh}\left[N_\theta - \nu N_r\right] = 2f\gamma\sqrt{1+(\gamma)^2}\ \epsilon_\theta^o, \qquad 6.6.75$$

$$\gamma \frac{\partial u_\theta^o}{\partial \gamma} - u_\theta^o + \sqrt{1+(\gamma)^2}\ \frac{\partial u_r^o}{\partial \theta} = \frac{2f\gamma\sqrt{1+(\gamma)^2}}{Eh}\left[2(1+\nu)N_{r\theta}\right]$$
$$= 2f\gamma\sqrt{1+(\gamma)^2}\left[2\epsilon_{r\theta}^0\right]. \qquad 6.6.76$$

If the shell is closed at the apex or has a free edge, the form of the solutions for the stress-resultants (cf. equations 6.6.71, 6.6.72, 6.6.73) and boundary conditions on u_r^o and u_θ^o (cf. equations 6.2.7 and 6.2.8) suggest that the solutions for the displacements should take on the forms,

$$u_r^o(\gamma,\theta) = m_0(\gamma) + m_1(\gamma)\sin\theta, \qquad 6.6.77$$

$$u_\theta^o(\gamma,\theta) = n_1(\gamma)\cos\theta, \qquad 6.6.78$$

$$w(\gamma,\theta) = p_0(\gamma) + p_1(\gamma)\sin\theta. \qquad 6.6.79$$

The proof goes through as that for the stress resultants. (cf. pp 33-37)

The higher harmonic terms which were omitted from equations (6.6.77), (6.6.78) and (6.6.79) correspond to displacements without straining the middle surface of the shell and are often referred to as the inextensional deformations. Thus we may conclude that shells with the given displacement conditions exhibit no inextensional deformations (cf. page 7).

Equations (6.6.77) through (6.6.79) hold also for the case where the shell is fixed tangentially at both edges. Here the inextensional

deformations must be of the form

$$u_{rn}(r,\theta) = m_n(r) \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}$$

$$= \left\{ c_1 f_1(r,n) + c_2 f_2(r,n) + c_3 f_3(r,n) + c_4 f_4(r,n) \right\} \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}$$

$$u_{\theta n}(r,\theta) = n_n(r) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$$

$$= \left\{ c_1 g_1(r,n) + c_2 g_2(r,n) + c_3 g_3(r,n) + c_4 g_4(r,n) \right\} \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}.$$

The constants of integration C_1, C_2, C_3 and C_4 are to be determined by the displacement boundary conditions (6.2.1) through (6.2.4). Thus, we have

$$c_1 f_1(r_1,n) + c_2 f_2(r_1,n) + c_3 f_3(r_1,n) + c_4 f_4(r_1,n) = 0,$$

$$c_1 f_1(r_2,n) + c_2 f_2(r_2,n) + c_3 f_3(r_2,n) + c_4 f_4(r_2,n) = 0,$$

$$c_1 g_1(r_1,n) + c_2 g_2(r_1,n) + c_3 g_3(r_1,n) + c_4 g_4(r_1,n) = 0,$$

$$c_1 g_1(r_2,n) + c_2 g_2(r_2,n) + c_3 g_3(r_2,n) + c_4 g_4(r_2,n) = 0.$$

Since $m_n(r)$ and $n_n(r)$ are linearly independent, the set of linear algebraic equations for C_1, C_2, C_3 and C_4 admits no non-trivial solutions.

Substitution of the expressions (6.6.77) through (6.6.79) into the system of partial differential equations 6.6.74, 6.6.75 and 6.6.76 yields the following set of five ordinary differential equations for the five unknown functions m_0, m_1, n_1, p_0 and p_1 (primes indicate

differentiation with respect to γ):

$$m_0' - \frac{p_0}{1+(\gamma)^2} = 2f\sqrt{1+(\gamma)^2}\ \epsilon_r^{*}, \qquad 6.6.80$$

$$m_0 - \gamma p_0 = 2f\gamma\sqrt{1+(\gamma)^2}\ \epsilon_\theta^{*}, \qquad 6.6.81$$

$$m_1' - \frac{p_1}{1+(\gamma)^2} = 2f\sqrt{1+(\gamma)^2}\ \epsilon_r^{(1)}, \qquad 6.6.82$$

$$-\sqrt{1+(\gamma)^2}\ n_1 + m_1 - \gamma p_1 = 2f\gamma\sqrt{1+(\gamma)^2}\ \epsilon_\theta^{(1)}, \qquad 6.6.83$$

$$\gamma n_1' - n_1 + \sqrt{1+(\gamma)^2}\ m_1 = 4f\gamma\sqrt{1+(\gamma)^2}\ \epsilon_{r\theta}^{(1)} \qquad 6.6.84$$

where for convenience we have separated the middle surface strains into components, defined as follows:

$$\epsilon_r^{0}(\gamma,\theta) = \epsilon_r^{*}(\gamma) + \epsilon_r^{(1)}(\gamma)\ \sin\ \theta\ , \qquad 6.6.85$$

$$\epsilon_\theta^o(\gamma,\theta) = \epsilon_\theta^*(\gamma) + \epsilon_\theta^{(1)}(\gamma)\sin\theta,$$

6.6.86

$$\epsilon_{r\theta}^o(\gamma,\theta) = \epsilon_{r\theta}^{(1)}\cos\theta.$$

6.6.87

It was noted, in a previous paragraph in this section, that the solutions for the force-resultants N_r and N_θ contained an axi-symmetric portion identical in form (except for a multiplicative factor of $\cos\psi$) to the solutions determined in section 6.4. Thus the results of section 6.4, can be used to determine the solutions for $m_0(\gamma)$ and $p_0(\gamma)$, the axi-symmetric portions of the solutions for the displacements u_r^o and w. There results (compare equations 6.4.20, 6.4.18 with equations 6.6.81 and 6.6.82):

$$m_0(\gamma) = \frac{4f^2 p_0 \cos\psi}{3E\sqrt{1+(\gamma)^2}}\left\{(1+\nu)\left(\gamma \ln\gamma - \frac{1}{\gamma}\right) + \left(\frac{3-\nu}{2}\right)(\gamma)^3 + \frac{1}{4}(\gamma)^5\right.$$
$$\left. + C_1\left[-(1+\nu)\gamma\ln\left(\frac{1+\sqrt{1+(\gamma)^2}}{\gamma}\right) + \sqrt{1+(\gamma)^2}\left(\gamma - \frac{1+\nu}{\gamma}\right)\right] + C_2\gamma\right\},$$

6.6.88

$$p_0(\gamma) = \frac{4f^2\rho_0 \cos\psi}{3E\sqrt{1+(\gamma)^2}} \left\{ (1+\nu)\left(\ln\gamma - \frac{1}{(\gamma)^2}\right) + \left(\frac{3-\nu}{2}\right)(\gamma)^2 + \frac{1}{4}(\gamma)^5 \right.$$

$$+ (1+(\gamma)^2)\left[2(\nu-1) + \frac{1+\nu}{(\gamma)^2} + \nu(\gamma)^2\right] + C_1\left[-(1+\nu)\ln\left(\frac{1+\sqrt{1+(\gamma)^2}}{\gamma}\right)\right.$$

$$\left.\left. + \sqrt{1+(\gamma)^2}\,(1+\nu)\right] + C_2 \right\} \qquad 6.6.89$$

where C_1 is the constant of integration arisen from the integration of force equilibrium equation and C_2 is an additional constant from the integration of equation 6.6.80.

The three remaining equations, 6.6.82, 6.6.83, 6.6.84, which determine the asymmetric component of the displacements can be rearranged to yield

$$\left(\frac{n_1'}{\gamma}\right)' = \frac{2f}{\gamma}\left\{\left[\sqrt{1+(\gamma)^2}\,2\epsilon_{r\theta}^{(1)}\right]' + \frac{\epsilon_\theta^{(1)} - \epsilon_r^{(1)}}{\gamma} - \gamma\epsilon_r^{(1)}\right\}, \qquad 6.6.90$$

$$m_1 = 4f\gamma\,\epsilon_{r\theta}^{(1)} - \frac{\gamma}{\sqrt{1+(\gamma)^2}}\,n_1' + \frac{n_1}{\sqrt{1+(\gamma)^2}}, \qquad 6.6.91$$

$$p_1 = -2f\,(1+(\gamma)^2)^{3/2}\,\epsilon_r^{(1)} + (1+(\gamma)^2)\,m_1'. \qquad 6.6.92$$

We can obtain the solution for n_1 by a simple, though tedious, double quadrature

$$n_1(\gamma) = 2f\left\{\int^{\gamma}\left[\gamma\int^{\gamma}\left(\frac{\left[\sqrt{1+(\gamma)^2}\,2\epsilon_{r\theta}^{(1)}\right]'}{\gamma} + \frac{\epsilon_{\theta}^{(1)}-\epsilon_r^{(1)}}{(\gamma)^2} - \epsilon_r^{(1)}\right)\partial\gamma\right]d\gamma\right.$$
$$\left. + \frac{c_5(\gamma)^2}{2} + c_6\right\} \qquad 6.6.93$$

or $$n_1(\gamma) = \frac{4f\rho_0\sin\psi}{E}\left\{C_6 + \frac{c_5(\gamma)^2}{2} + C_4 f_2(\gamma) + C_3 f_1(\gamma) + f_3(\gamma)\right\}$$

6.6.94

where

$$f_1(\gamma) = \frac{(1+(\gamma)^2)^{3/2}}{6} - (1+\nu)\left[\sqrt{1+(\gamma)^2} + \ell n\left(\sqrt{1+(\gamma)^2}-1\right) - \ell n(\gamma)\right], \qquad 6.6.95$$

$$f_2(\gamma) = \frac{(1-\nu)}{4}(\gamma)^2\,\ell n\left[\frac{\sqrt{1+(\gamma)^2}-1}{\gamma}\right] + \frac{(1-\nu)}{4}\left[1+\sqrt{1+(\gamma)^2}\right] - \frac{(1+\nu)}{2}\,\frac{\left[1+(\gamma)^2\right]^{3/2}}{(\gamma)^2}, \qquad 6.6.96$$

$$f_3(\gamma) = \frac{(\gamma)^6}{360} - \frac{(40+29\nu)}{240}(\gamma)^4 - \frac{(1+\nu)}{30(\gamma)^2} - \frac{(1-\nu)(\gamma)^2}{120} + \ell n(\gamma)\left[\frac{(1-\nu)}{60}(\gamma)^2\right.$$
$$\left. - \frac{(1+\nu)}{3}\right], \qquad 6.6.97$$

c_5 and c_6 are constants of integration.

By equations 6.6.91 and 6.6.83, we have also

$$m_1(\gamma) = \frac{8f^2\rho \sin\psi}{E}\left\{C_3 f_4(\gamma) + C_4 f_5(\gamma) - \frac{C_5(\gamma)^2}{2\sqrt{1+(\gamma)^2}} + \frac{C_6}{\sqrt{1+(\gamma)^2}} + f_6(\gamma)\right\} \qquad 6.6.98$$

and

$$p_1(\gamma) = \frac{8f^2\rho \sin\psi}{E}\left\{C_3 f_7(\gamma) + C_4 f_8(\gamma) + C_5 f_9(\gamma) + C_6 f_{10}(\gamma) + f_{11}(\gamma)\right\} \qquad 6.6.99$$

where

$$f_4(\gamma) = \frac{f_1(\gamma)}{\sqrt{1+(\gamma)}^2} - \frac{(\gamma)^2}{2}, \qquad 6.6.100$$

$$f_5(\gamma) = \frac{f_2(\gamma)}{\sqrt{1+(\gamma)^2}} + \frac{(1+\nu)}{(\gamma)^2}\left(\frac{(\gamma)^2}{2} + 1\right) - \frac{(1-\nu)(\gamma)^2}{2\sqrt{1+(\gamma)^2}}\left[\ln\left(\frac{\sqrt{1+(\gamma)^2}-1}{\gamma}\right)\right] \qquad 6.6.101$$

$$f_6(\gamma) = \frac{f_3(\gamma)}{\sqrt{1+(\gamma)^2}} - \frac{\gamma}{\sqrt{1+(\gamma)^2}}\left[\frac{(1-\nu)}{30}\gamma \ln(\gamma) + \frac{(1+\nu)}{15(\gamma)^3} - \frac{(1+\nu)}{3\gamma} - \frac{(40+29\nu)}{60}(\gamma)^3 + \frac{(\gamma)^5}{60}\right] + \frac{2(1+\nu)\left[1-4(\gamma)^2\right]\left[1+(\gamma)^2\right]^{3/2}}{15(\gamma)^2} \qquad 6.6.102$$

$$f_7(\gamma) = \frac{1}{\gamma}\left[f_4(\gamma) - \sqrt{1+(\gamma)^2}\, f_1(\gamma) - \frac{(1+\nu)}{2} - \frac{\nu(\gamma)^2}{2}\right], \qquad 6.6.103$$

$$f_8(\gamma) = \frac{1}{\gamma}\left[f_5(\gamma) - \sqrt{1+(\gamma)^2}\; f_2(\gamma) - \frac{1}{(\gamma)^2}\,(1+\nu) - \nu \right], \tag{6.6.104}$$

$$f_9(\gamma) = \frac{-\frac{1}{2}(\gamma)}{\sqrt{1+(\gamma)^2}}\left[2+(\gamma)^2 \right], \tag{6.6.105}$$

$$f_{10}(\gamma) = \frac{-\gamma}{\sqrt{1+(\gamma)^2}}, \tag{6.6.106}$$

$$f_{11}(\gamma) = \frac{1}{\gamma}\left[f_6(\gamma) - \sqrt{1+(\gamma)^2}\; f_3(\gamma) - \frac{\sqrt{1+(\gamma)^2}}{(\gamma)^2}\left\{ \frac{(\gamma)^4}{2} + \frac{[1+(\gamma)^2]^2}{15} + \frac{\nu[1+(\gamma)^2]^3}{15} \right\} \right]. \tag{6.6.107}$$

Several different combinations of boundary conditions are considered and numerical calculations have been made. It is evident that a complete description of the deformed shell for each orientation (i.e., each value of ψ) would require an inordinate number of curves and tables. In lieu of presenting such a mass of numerical data only the anti-symmetric part of the solution has been presented ($\psi = \frac{\pi}{2}$). The behavior of the shell for other values of ψ can be obtained by appropriate combinations of the results for $\psi = 0$ and $\psi = \frac{\pi}{2}$. The solutions for the force resultants and displacements can be cast into the forms (see equations 6.6.19, 6.6.20, 6.6.21, 6.6.77. 6.6.78, 6.6.79,

6.6.69, 6.6.70, 6.6.71, 6.6.72, 6.6.73, 6.6.94, 6.6.98, and 6.6.99),

$$N_r^* = \frac{N_r}{2f\rho_o h} = a_c^* \cos\psi + a_1^* \sin\theta \sin\psi \qquad 6.6.108$$

$$N_\theta^* = \frac{N_\theta}{2f\rho_o h} = b_o^* \cos\psi + b_1^* \sin\theta \sin\psi \qquad 6.6.109$$

$$N_{r\theta}^* = c_1^* \cos\theta \sin\psi \qquad 6.6.110$$

$$u_r^* = \frac{E u_r^o}{4f^2 \rho_o} = m_o^* \cos\psi + m_1^* \sin\theta \sin\psi \qquad 6.6.111$$

$$u_\theta^* = \frac{E u_\theta^o}{4f^2 \rho_o} = n_1^* \cos\theta \sin\psi \qquad 6.6.112$$

$$W^* = \frac{E w}{4f^2 \rho_o} = p_c^* \cos\psi + p_1^* \sin\theta \sin\psi \qquad 6.6.113$$

The a_o^*, b_c^*, m_o^*, p_o^* are the non-dimensionalized symmetric parts of the solution.

$$a_o^* = \frac{a_o}{2f\rho_o h \cos\psi} \qquad 6.6.114$$

$$b_o^* = \frac{b_o}{2f\rho_o h \cos\psi} \qquad 6.6.115$$

$$m_o^* = \frac{E m_o}{4f^2 \rho_o \cos\psi} \qquad 6.6.116$$

$$p_o^* = \frac{E p_o}{4f^2 \rho_o \cos\psi} \qquad 6.6.117$$

These have already been plotted in the previous sections of this chapters for the case of $\psi = 0$. The other starred quantities are the asymmetric part of the solution and are defined as follows:

$$a_1^* = \frac{E a_1}{2f\rho_o h \sin\psi} \qquad 6.6.118$$

$$b_1^* = \frac{E b_1}{2 f \rho_o h \sin\psi} \qquad 6.6.119$$

$$c_1^* = \frac{E c_1}{2 f \rho_o h \cos\psi} \qquad 6.6.120$$

$$m_1^* = \frac{E m_1}{4 f^2 \rho_o \sin\psi} \qquad 6.6.121$$

$$n_1^* = \frac{E n_1}{4 f^2 \rho_o \sin\psi} \qquad 6.6.122$$

$$p_1^* = \frac{E p_1}{4 f^2 \rho_o \sin\psi} \qquad 6.6.123$$

The results which are presented in the following sections are the above six functions for the case of $\psi = \frac{\pi}{2}$. Thus, the results represent the completely anti-symmetric behavior $N_r^*, N_\theta^*, N_{r\theta}^*, u_r^*, u_\theta^*$, and w^*. Additionally, it is seen that a_1^*, b_1^*, m_1^* and p_1^* represent the stress resultants and displacements along the bottom vertical radius ($\theta = \frac{\pi}{2}$) whereas c_1^* and n_1^* represent $N_{r\theta}^*$ and u_θ^* along the horizontal radius ($\theta = 0$). The values at other angular positions (i.e., other values of θ) can be obtained by multiplying the given values by $\sin\theta$ or $\cos\theta$, whichever is appropriate.

6.6.1 BOTH EDGES RESTRAINED IN THE TANGENTIAL DIRECTIONS

The condition of vanishing tangential displacements at the edges requires that

$$C_3 f_1(\gamma_1) + C_4 f_2(\gamma_1) + C_5 \frac{(\gamma_1)^2}{2} + C_6 = -f_3(\gamma_1) , \qquad 6.6.1.1$$

$$c_3 f_1(\gamma_2) + c_4 f_4(\gamma_2) + \frac{c_5(\gamma_2)^2}{2} + c_6 = -f_3(\gamma_2) \qquad 6.6.1.2$$

$$c_3 f_4(\gamma_1) + c_4 f_5(\gamma_1) - \frac{c_5(\gamma_1)^2}{2\sqrt{1+(\gamma_1)^2}} + \frac{c_6}{\sqrt{1+(\gamma_1)^2}} = -f_6(\gamma_1) \qquad 6.6.1.3$$

$$c_3 f_4(\gamma_2) + c_4 f_5(\gamma_2) - \frac{c_5(\gamma_2)^2}{2\sqrt{1+(\gamma_2)^2}} + \frac{c_6}{\sqrt{1+(\gamma_2)^2}} = -f_6(\gamma_2) \qquad 6.6.1.4$$

All $f_i(\gamma)$ have been defined previously. The system of equations (6.6.1.1) through (6.6.1.4) can be solved simultaneously for C_3, C_4 C_5 and C_6.

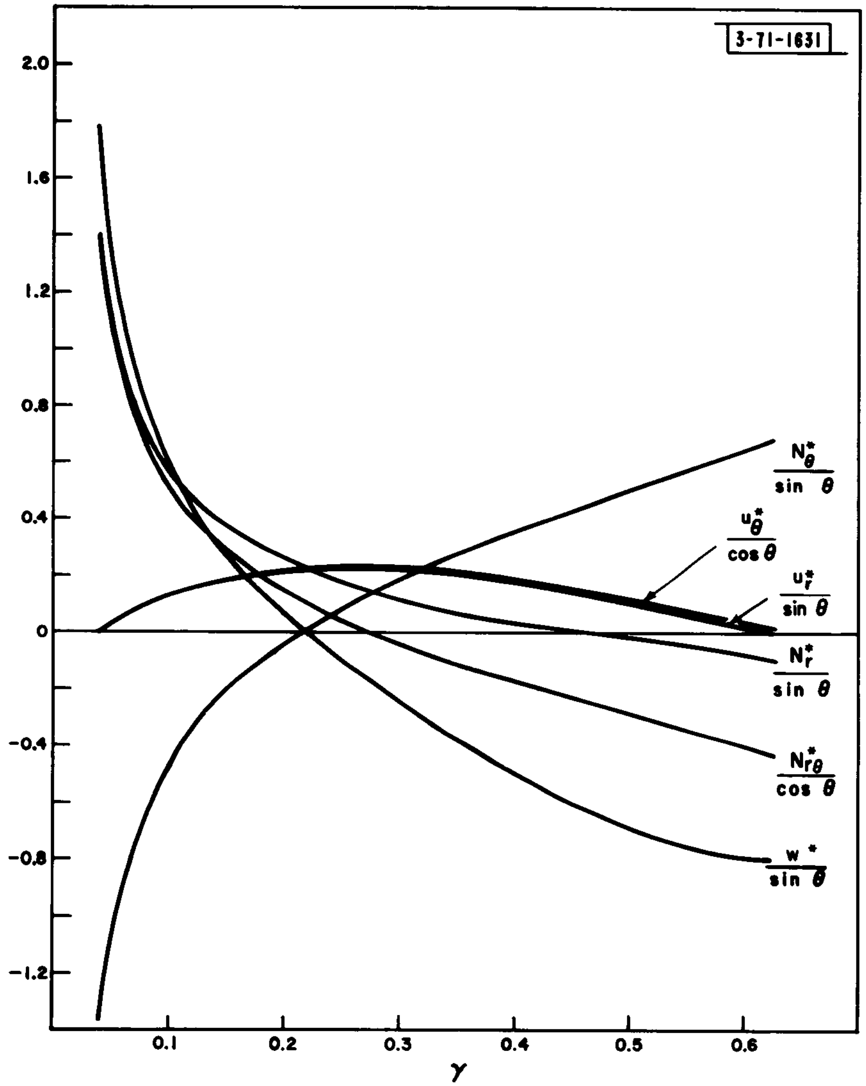
Six different cases have been analyzed and the results are presented in curve form as well as in tabular form. It should be observed that tangential displacements, as represented by m_1^* and n_1^*, become of the same order of magnitude as the normal displacement (represented by ρ_1^*) over a portion of the shell. This is in contrast to the symmetric behavior in which the tangential displacement is generally an order of magnitude smaller than the normal displacement (see section 6.4). Also, the largest value of ρ_1^* (which is w^* at $\theta = \frac{\pi}{2}$ and $\psi = \frac{\pi}{2}$) is larger than w^* for the symmetric case (compare case 6.6.1.2 with 6.4.1.2 and case 6.6.1.4 with 6.4.1.4). This is a bit surprising until it is remembered that the value of W^* at the boundaries is determined primarily by the need to accommodate the membrane boundary conditions (see discussions in

sections 6.4.1, 6.4.2, and 6.4.3).

The list of configurations which have been analyzed are presented in Table 6.6.1.

Table 6.6.1

Case	Restrained at γ_1	Restrained at γ_2
6.6.1.1	.0417	.6250
6.6.1.2	.0104	.6250
6.6.1.3	.0104	.3854
6.6.1.4	.3854	.6250
6.6.1.5	.0416	.7990
6.6.1.6	.0416	1.0243



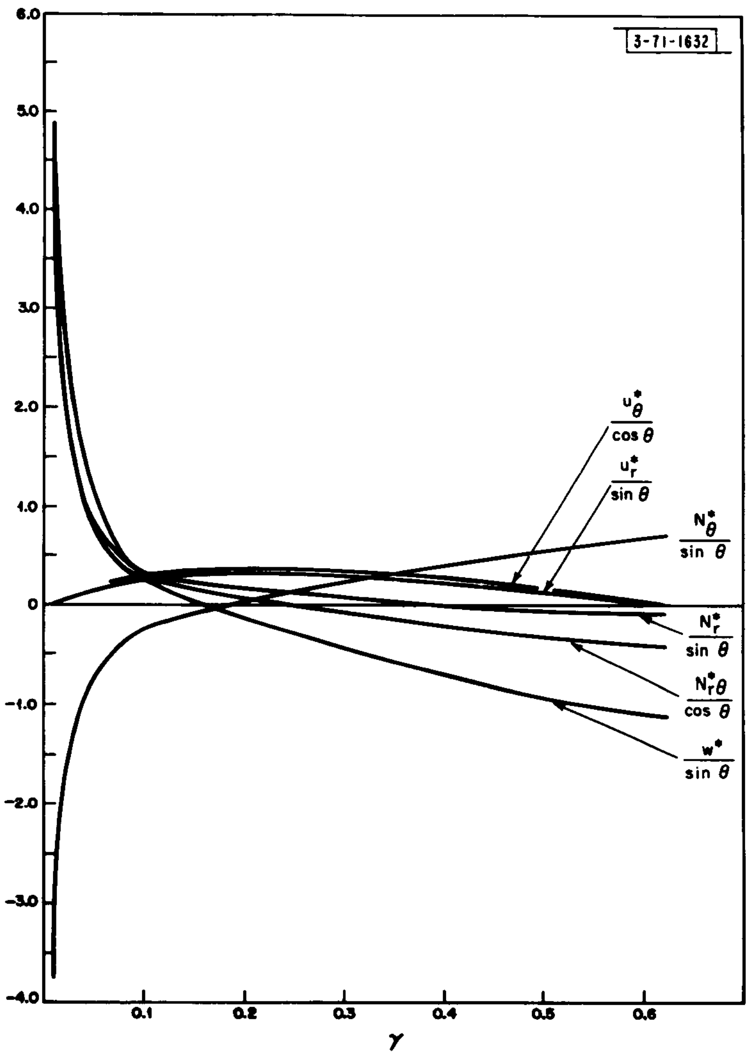
CASE 6.6.1.1 ASYMMETRIC GRAVITY

Restrained $\gamma_1 = 0.0417$

Restrained $\gamma_2 = 0.625$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0416	1.403	−1.359	1.390
0.0999	0.567	−0.462	0.515
0.1583	0.337	−0.170	0.253
0.2166	0.223	0.0036	0.107
0.2749	0.150	0.135	0.0042
0.3333	0.0972	0.245	−0.0803
0.3916	0.0541	0.344	−0.155
0.4499	0.0164	0.436	−0.224
0.5083	−0.0181	0.522	−0.291
0.5666	−0.0514	0.605	−0.356
0.6249	−0.0843	0.685	−0.422

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0416	−0.000	−0.000	1.781
0.0999	0.125	0.127	0.608
0.1583	0.178	0.184	0.224
0.2166	0.201	0.212	−0.0086
0.2749	0.205	0.221	−0.184
0.3333	0.194	0.216	−0.331
0.3916	0.172	0.198	−0.458
0.4499	0.140	0.168	−0.571
0.5083	0.100	0.125	−0.672
0.5666	0.0535	0.0697	0.760
0.6249	0.000	0.000	−0.838



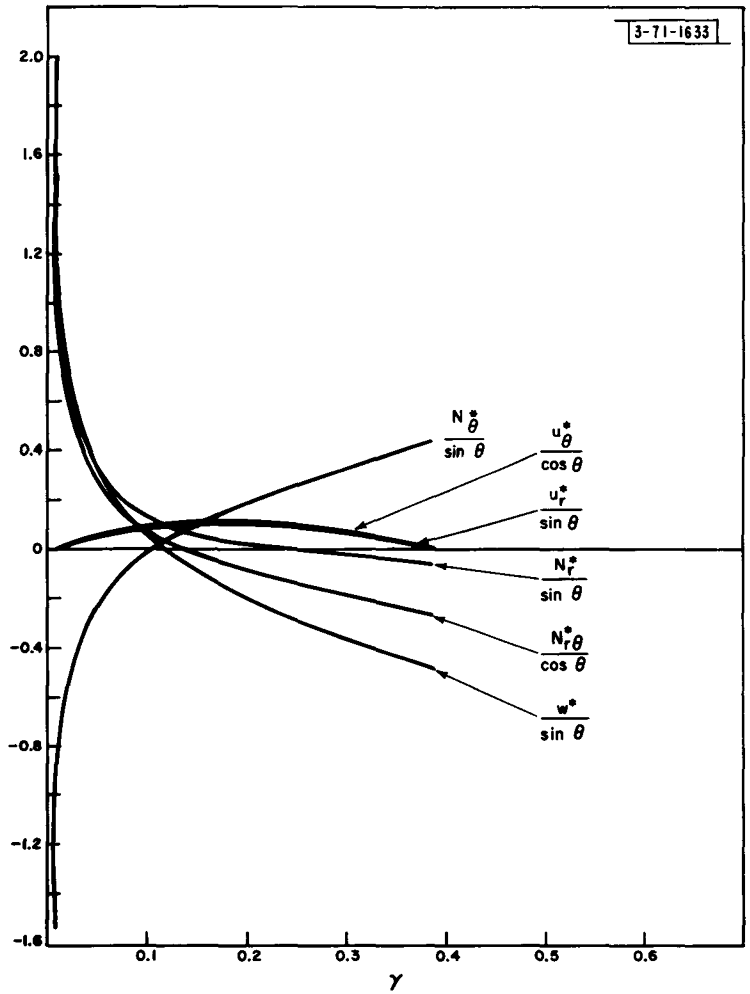
CASE 6.6.1.2 ASYMMETRIC GRAVITY

Restrained $\gamma_1 = 0.0104$

Restrained $\gamma_2 = 0.625$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0104	3.793	−3.783	3.806
0.0718	0.535	−0.460	0.497
0.1333	0.266	−0.128	0.196
0.1947	0.157	0.043	0.0559
0.2562	0.0928	0.169	−0.0409
0.3177	0.0462	0.275	−0.120
0.3791	0.0080	0.372	−0.190
0.4406	−0.0258	0.462	−0.257
0.5020	−0.0578	0.548	−0.322
0.5635	−0.0892	0.631	−0.387
0.6249	−0.120	0.711	−0.453

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0104	−0.000	−0.000	4.930
0.0718	0.193	0.194	0.597
0.1333	0.242	0.247	0.158
0.1947	0.257	0.267	−0.0727
0.2562	0.253	0.269	−0.243
0.3177	0.234	0.257	−0.385
0.3791	0.204	0.232	−0.510
0.4406	0.164	0.194	−0.621
0.5020	0.116	0.143	−0.720
0.5635	0.0615	0.0791	−0.807
0.6249	−0.000	−0.000	−0.882



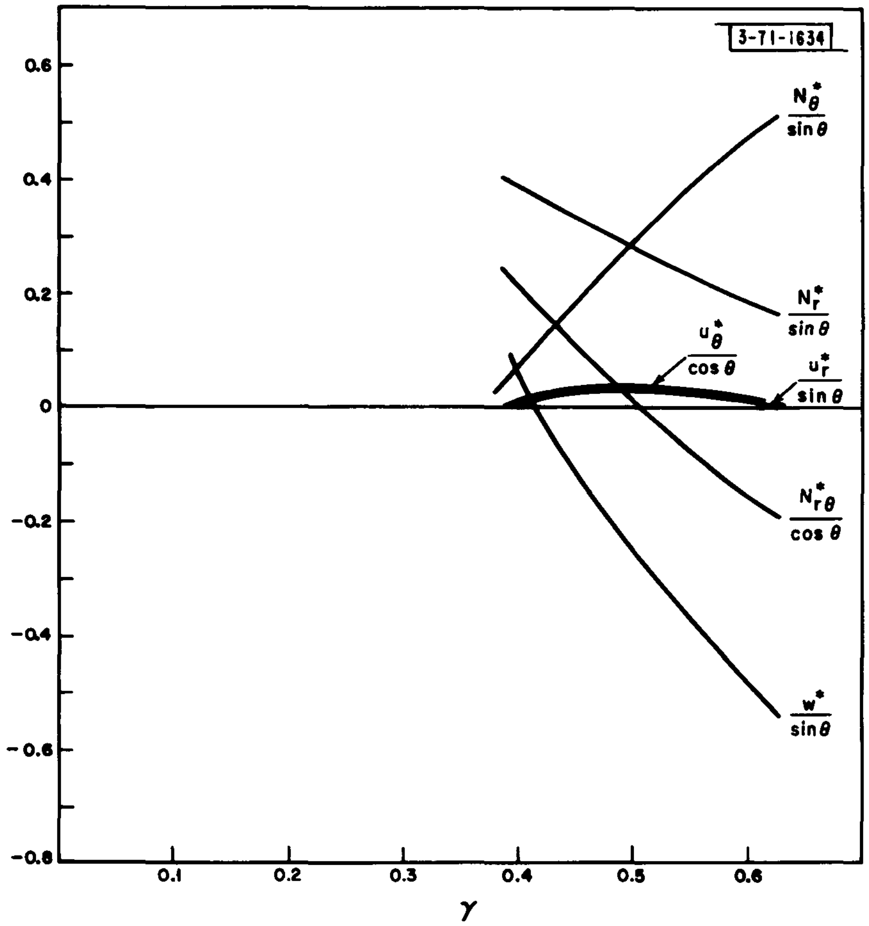
CASE 6.6.1.3 ASYMMETRIC GRAVITY

Restrained $\gamma_1 = 0.0104$

Restrained $\gamma_2 = 0.3854$

γ	$\frac{N^*_r}{\sin\theta}$	$\frac{N^*_\theta}{\sin\theta}$	$\frac{N^*_{r\theta}}{\cos\theta}$
0.0104	1.544	-1.535	1.558
0.0479	0.327	-0.278	0.302
0.0854	0.169	-0.0826	0.125
0.1229	0.102	0.0224	0.0394
0.1604	0.0616	0.100	-0.0199
0.1979	0.0328	0.166	-0.0680
0.2354	0.0098	0.226	-0.110
0.2729	-0.0099	0.282	-0.149
0.3104	-0.0277	0.335	-0.186
0.3479	-0.0445	0.387	-0.222
0.3854	-0.0606	0.438	-0.258

γ	$\frac{u^*_r}{\sin\theta}$	$\frac{u^*_\theta}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0104	-0.000	-0.000	2.0087
0.0479	0.0624	0.0626	0.371
0.0854	0.0815	0.0822	0.122
0.1229	0.0885	0.0899	-0.0090
0.1604	0.0884	0.0908	-0.104
0.1979	0.0831	0.0864	-0.184
0.2354	0.0734	0.0774	-0.255
0.2729	0.0599	0.0642	-0.319
0.3104	0.0429	0.0469	-0.380
0.3479	0.0229	0.0255	-0.436
0.3854	-0.000	-0.000	-0.489



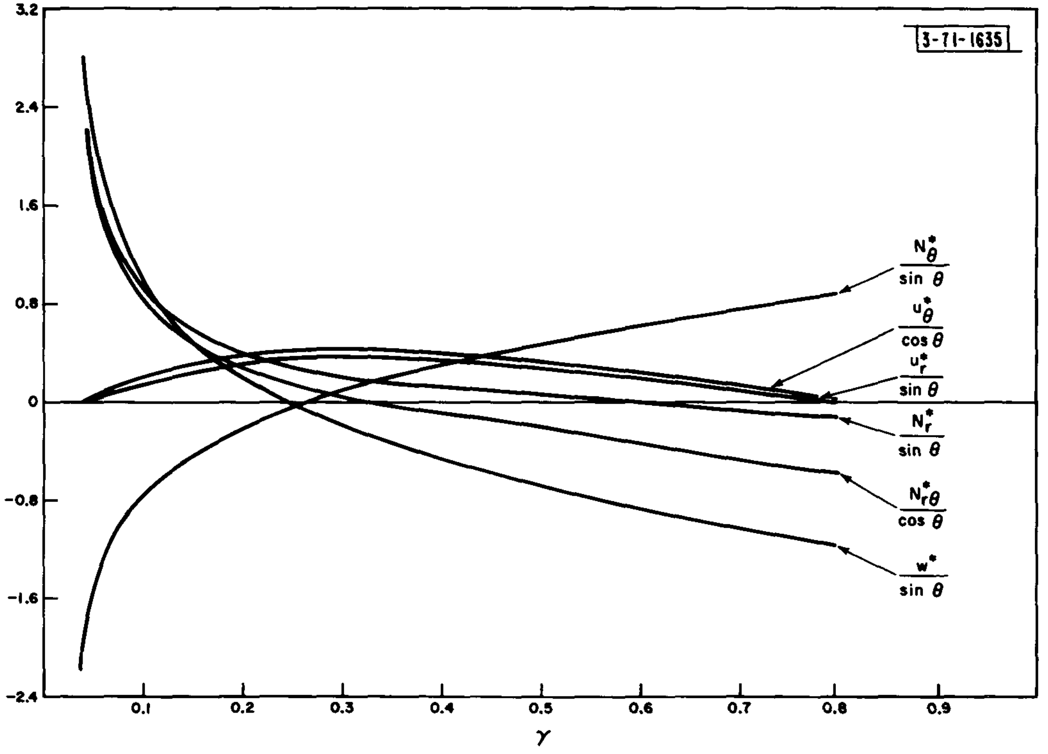
CASE 6.6.1.4 ASYMMETRIC GRAVITY

Restrained $\gamma_1 = 0.3854$

Restrained $\gamma_2 = 0.625$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.3854	0.403	0.0339	0.241
0.4093	0.373	0.0891	0.186
0.4333	0.345	0.142	0.134
0.4572	0.319	0.192	0.0868
0.4812	0.294	0.241	0.0415
0.5052	0.270	0.289	-0.0014
0.5291	0.248	0.335	-0.0426
0.5531	0.226	0.379	-0.0823
0.5770	0.205	0.422	-0.120
0.6010	0.185	0.465	-0.158
0.6249	0.165	0.506	-0.194

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.3854	-0.000	-0.000	0.0934
0.4093	0.0102	0.0140	0.0129
0.4333	0.0178	0.0246	-0.0626
0.4572	0.0228	0.0320	-0.133
0.4812	0.0255	0.0363	-0.201
0.5052	0.0259	0.0376	-0.265
0.5291	0.0244	0.0360	-0.325
0.5531	0.0209	0.0314	-0.383
0.5770	0.0156	0.0238	-0.437
0.6010	0.0086	0.0134	-0.489
0.6249	-0.000	0.000	-0.538



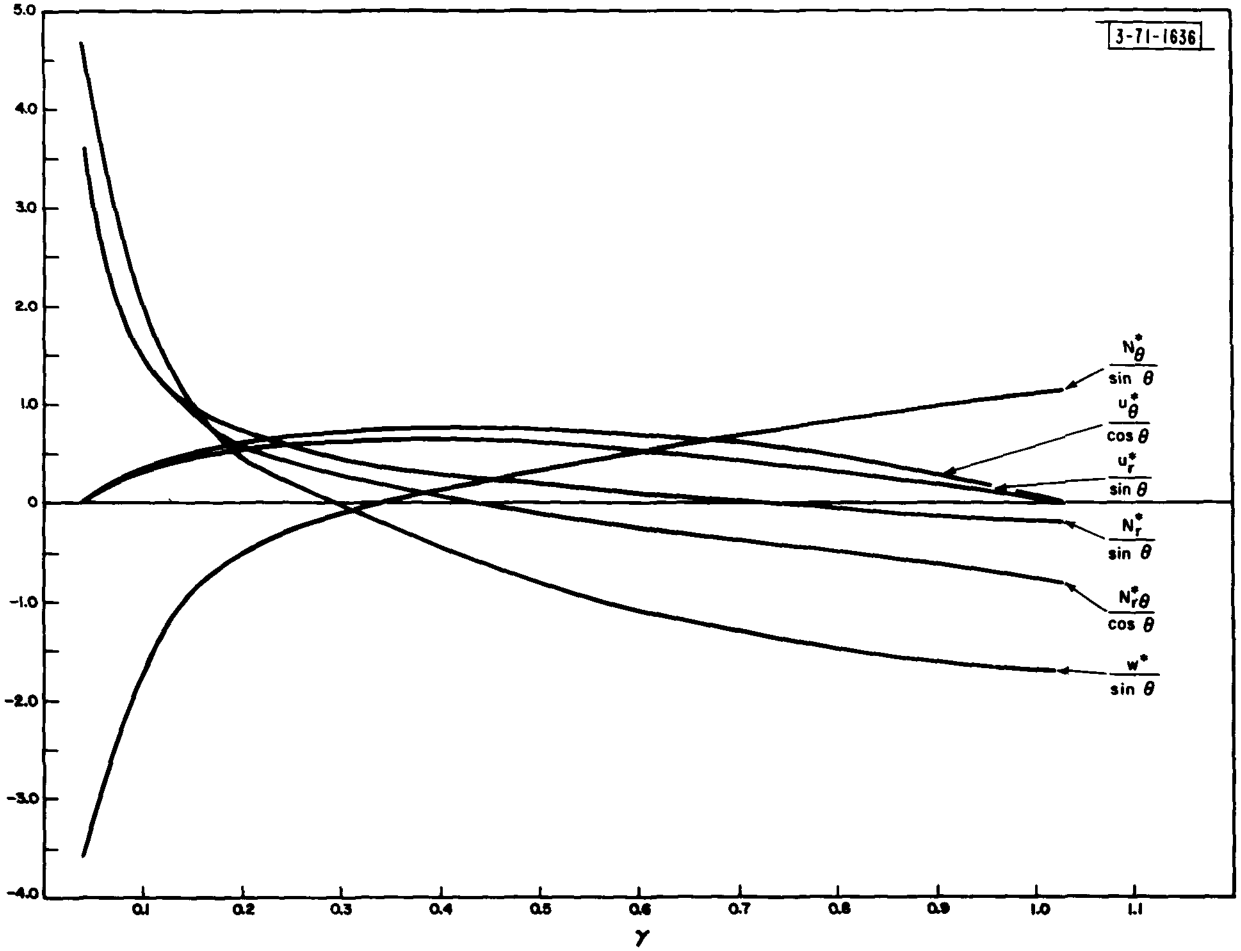
CASE 6.6.1.5 ASYMMETRIC GRAVITY

Restrained $\gamma_1 = 0.0416$

Restrained $\gamma_2 = 0.799$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0416	2.2053	−2.159	2.198
0.1173	0.764	−0.636	0.700
0.1930	0.438	−0.230	0.333
0.2687	0.286	0.001	0.138
0.3444	0.191	0.173	0.002
0.4201	0.121	0.316	−0.109
0.4958	0.0645	0.444	−0.209
0.5715	0.013	0.561	−0.303
0.6472	−0.0351	0.672	−0.395
0.7229	−0.0833	0.777	−0.486
0.7986	−0.132	0.879	−0.579

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0416	−0.000	−0.000	2.823
0.1173	0.234	0.239	0.815
0.1930	0.324	0.339	0.259
0.2687	0.360	0.389	−0.0708
0.3444	0.362	0.406	−0.320
0.4201	0.339	0.399	−0.526
0.4958	0.297	0.368	−0.703
0.5715	0.240	0.314	−0.855
0.6472	0.170	0.236	−0.985
0.7229	0.0900	0.132	−1.0925
0.7986	−0.000	−0.000	−1.176



CASE 6.6.1.6 ASYMMETRIC GRAVITY

Restrained $\gamma_1 = 0.0416$

Restrained $\gamma_2 = 1.0243$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0416	3.599	−3.551	3.601
0.1399	1.052	−0.892	0.972
0.2381	0.589	−0.319	0.451
0.3364	0.382	−0.0071	0.188
0.4347	0.256	0.219	0.0043
0.5329	0.162	0.406	−0.146
0.6312	0.0840	0.571	−0.283
0.7295	0.0112	0.722	−0.414
0.8277	−0.0606	0.863	−0.545
0.9260	−0.134	0.998	−0.678
1.0243	−0.213	1.128	−0.814

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0416	−0.000	−0.000	4.634
0.1399	0.450	0.462	1.100
0.2381	0.605	0.644	0.271
0.3364	0.658	0.735	−0.220
0.4347	0.651	0.772	−0.593
0.5329	0.602	0.763	−0.898
0.6312	0.522	0.712	−1.153
0.7295	0.417	0.615	−1.361
0.8277	0.294	0.469	−1.525
0.9260	0.154	0.266	−1.641
1.0243	0.000	0.000	−1.706

6.6.2 ONE EDGE FREE AND ONE EDGE RESTRAINED IN THE TANGENTIAL DIRECTION

We shall specify the free edge to be γ_1 and the restrained edge to be γ_2. The constants of integration are

$$C_3 = \frac{-\left[1+(\gamma_1)^2\right]^{3/2}}{3} \tag{6.6.2.1}$$

$$C_4 = \frac{C_3(\gamma_1)^2}{2} - \frac{\left[1-4(\gamma_1)^2\right]\left[1+(\gamma_1)^2\right]^{3/2}}{15} \tag{6.6.2.2}$$

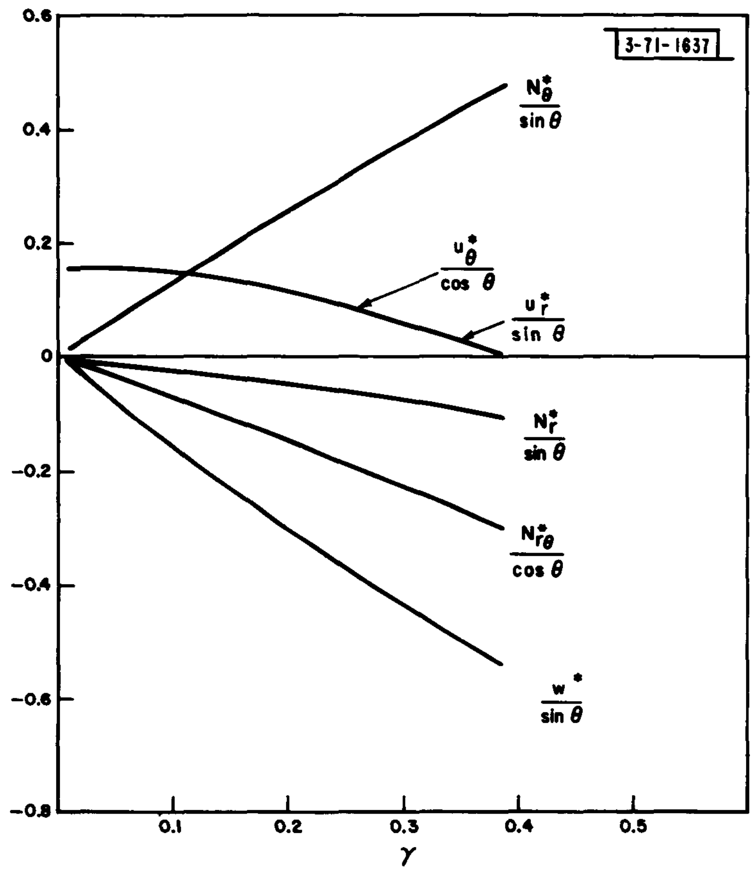
$$C_5 = \frac{1}{(\gamma_2)^2}\left[-f_3(\gamma_2) + \sqrt{1+(\gamma_2)^2}\, f_6(\gamma_2) + C_4\left\{f_2(\gamma_2) - \sqrt{1+(\gamma_2)^2}\, f_5(\gamma_2)\right\} + C_3\left\{f_1(\gamma_2) - \sqrt{1+(\gamma_2)^2}\, f_4(\gamma_2)\right\}\right] \tag{6.6.2.3}$$

$$C_6 = -\left\{f_3(\gamma_2) + \frac{C_5(\gamma_2)^2}{2} + C_4 f_2(\gamma_2) + C_3 f_1(\gamma_2)\right\} \tag{6.6.2.4}$$

The seven configurations which have been analyzed in this section are summarized in Table 6.6.2.1. The same discussion made in section 6.6.1 applies and no additional comments are required.

Table 6.6.2.1

Case	Free at γ_1	Restrained at γ_2
6.6.2.1	.0104	.3854
6.6.2.2	.0104	.6250
6.6.2.3	.0416	.6250
6.6.2.4	.6250	.3854
6.6.2.5	.0104	.9982
6.6.2.6	.1770	1.0243
6.6.2.7	.4010	1.0243



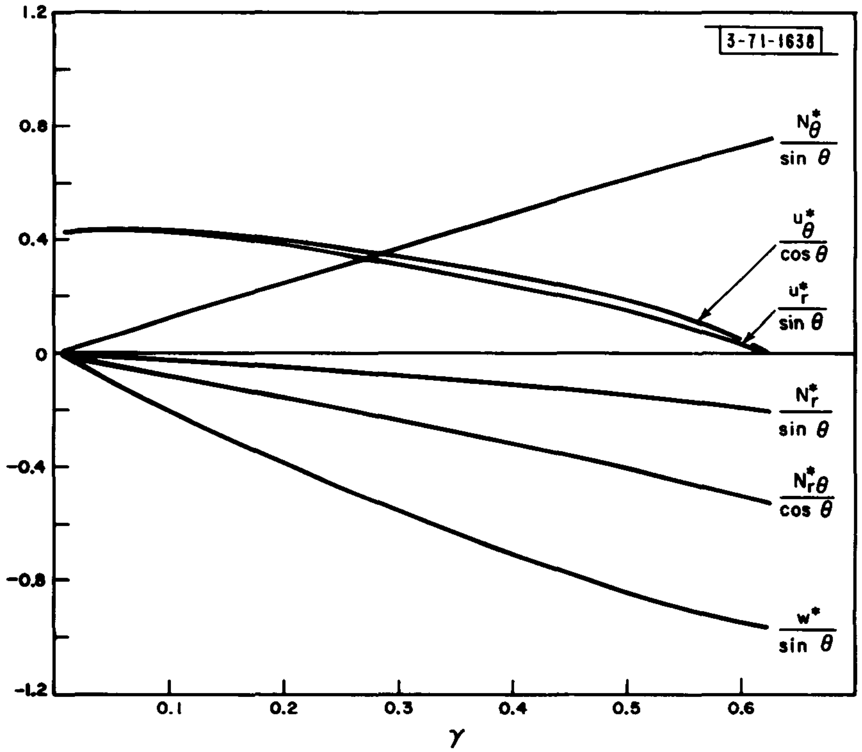
CASE 6.6.2.1 ASYMMETRIC GRAVITY

Free $\gamma_1 = 0.0104$

Restrained $\gamma_2 = 0.3854$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0104	0.0	0.0170	0.0065
0.0479	−0.0109	0.0588	−0.0347
0.0854	−0.0208	0.106	−0.0635
0.1229	−0.0306	0.153	−0.0921
0.1604	−0.0404	0.199	−0.120
0.1979	−0.0504	0.246	−0.149
0.2354	−0.0607	0.293	−0.179
0.2729	−0.0713	0.339	−0.208
0.3104	−0.0823	0.385	−0.238
0.3479	−0.0937	0.431	−0.269
0.3854	−0.105	0.477	−0.300

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0104	0.153	0.153	−0.0120
0.0479	0.151	0.151	−0.0721
0.0854	0.146	0.147	−0.129
0.1229	0.137	0.139	−0.186
0.1604	0.126	0.129	−0.242
0.1979	0.111	0.115	−0.297
0.2354	0.0946	0.0991	−0.350
0.2729	0.0747	0.0795	−0.402
0.3104	0.0522	0.0565	−0.452
0.3479	0.0272	0.0301	−0.499
0.3854	0.000	0.000	−0.545



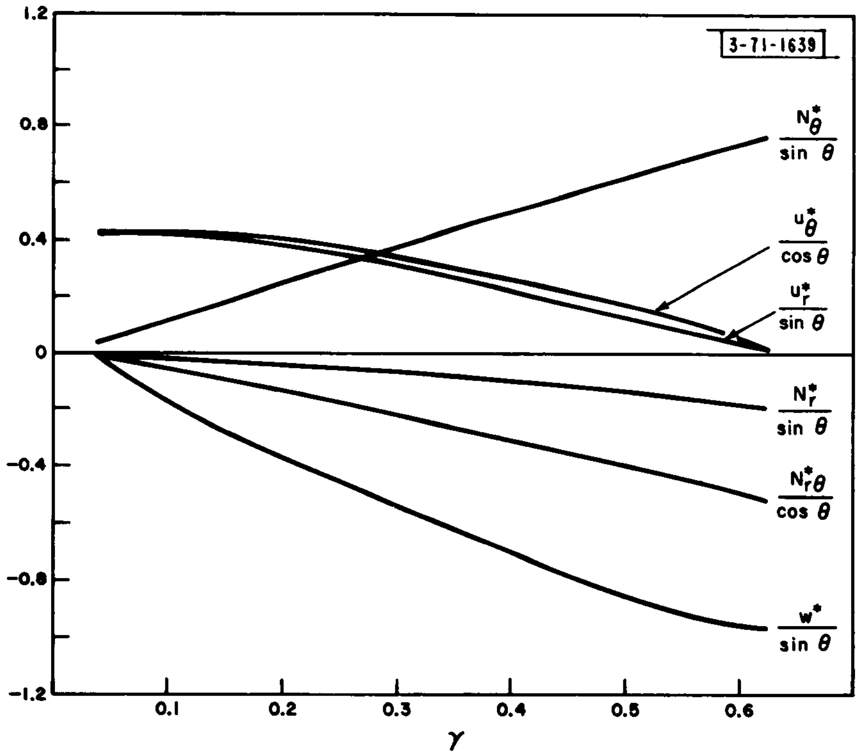
CASE 6.6.2.2 ASYMMETRIC GRAVITY

Free $\gamma_1 = 0.0104$

Restrained $\gamma_2 = 0.625$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0104	−0.0082	0.0170	0.0065
0.0718	−0.0173	0.0890	−0.0531
0.1333	−0.0333	0.166	−0.100
0.1947	−0.0496	0.242	−0.147
0.2562	−0.0666	0.318	−0.195
0.3177	−0.0845	0.394	−0.244
0.3791	−0.103	0.469	−0.295
0.4406	−0.124	0.544	−0.347
0.5020	−0.146	0.618	−0.401
0.5635	−0.169	0.692	−0.457
0.6249	−0.195	0.765	−0.516

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0104	0.431	0.431	−0.0158
0.0718	0.425	0.427	−0.135
0.1333	0.409	0.415	−0.250
0.1947	0.384	0.396	−0.362
0.2562	0.350	0.369	−0.470
0.3177	0.308	0.334	−0.573
0.3791	0.2591	0.289	−0.669
0.4406	0.202	0.235	−0.758
0.5020	0.140	0.169	−0.839
0.5635	0.0727	0.0915	−0.910
0.6249	−0.000	−0.000	−0.972



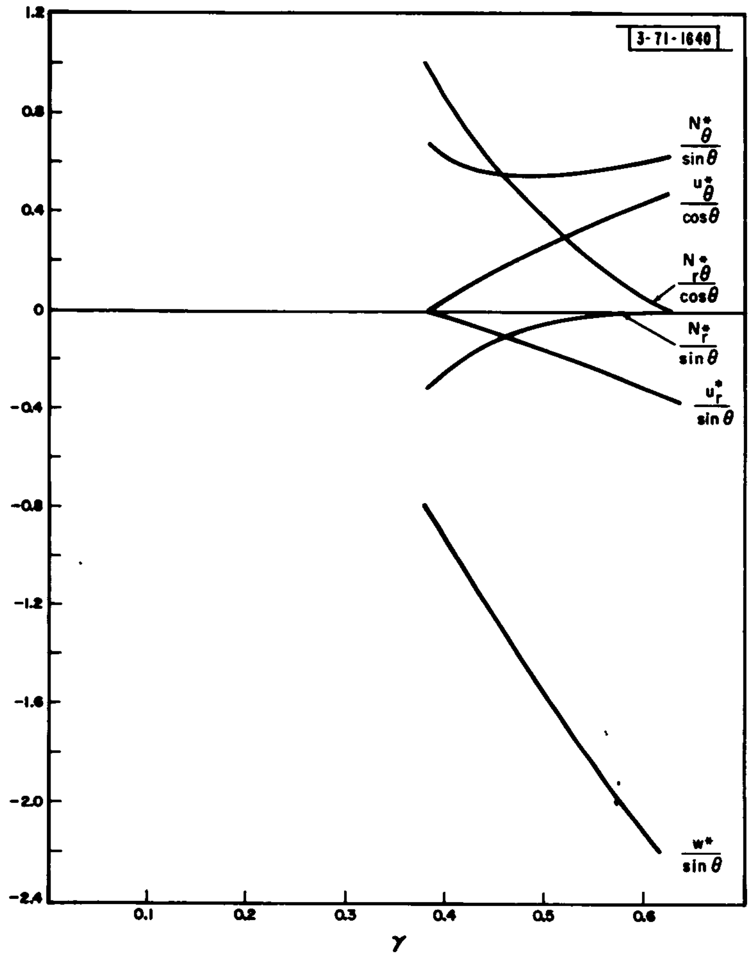
CASE 6.6.2.3 ASYMMETRIC GRAVITY

Free $\gamma_1 = 0.0416$

Restrained $\gamma_2 = 0.625$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0416	−0.000	0.0416	0.000
0.0999	−0.0171	0.117	−0.0657
* 0.1583	−0.0348	0.192	−0.113
0.2166	−0.0518	0.266	−0.160
0.2749	−0.0689	0.339	−0.207
0.3333	−0.0867	0.411	−0.254
0.3916	−0.105	0.483	−0.303
0.4499	−0.125	0.554	−0.353
0.5083	−0.146	0.624	−0.405
0.5666	−0.169	0.694	−0.459
0.6249	−0.194	0.764	−0.515

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0416	0.424	0.424	−0.0518
0.0999	0.415	0.419	−0.177
0.1583	0.397	0.405	−0.289
0.2166	0.371	0.385	−0.396
0.2749	0.336	0.358	−0.498
0.3333	0.295	0.322	−0.594
0.3916	0.247	0.278	−0.685
0.4499	0.193	0.225	−0.768
0.5083	0.133	0.161	−0.844
0.5666	0.0689	0.0870	−0.912
0.6249	−0.000	−0.000	−0.970



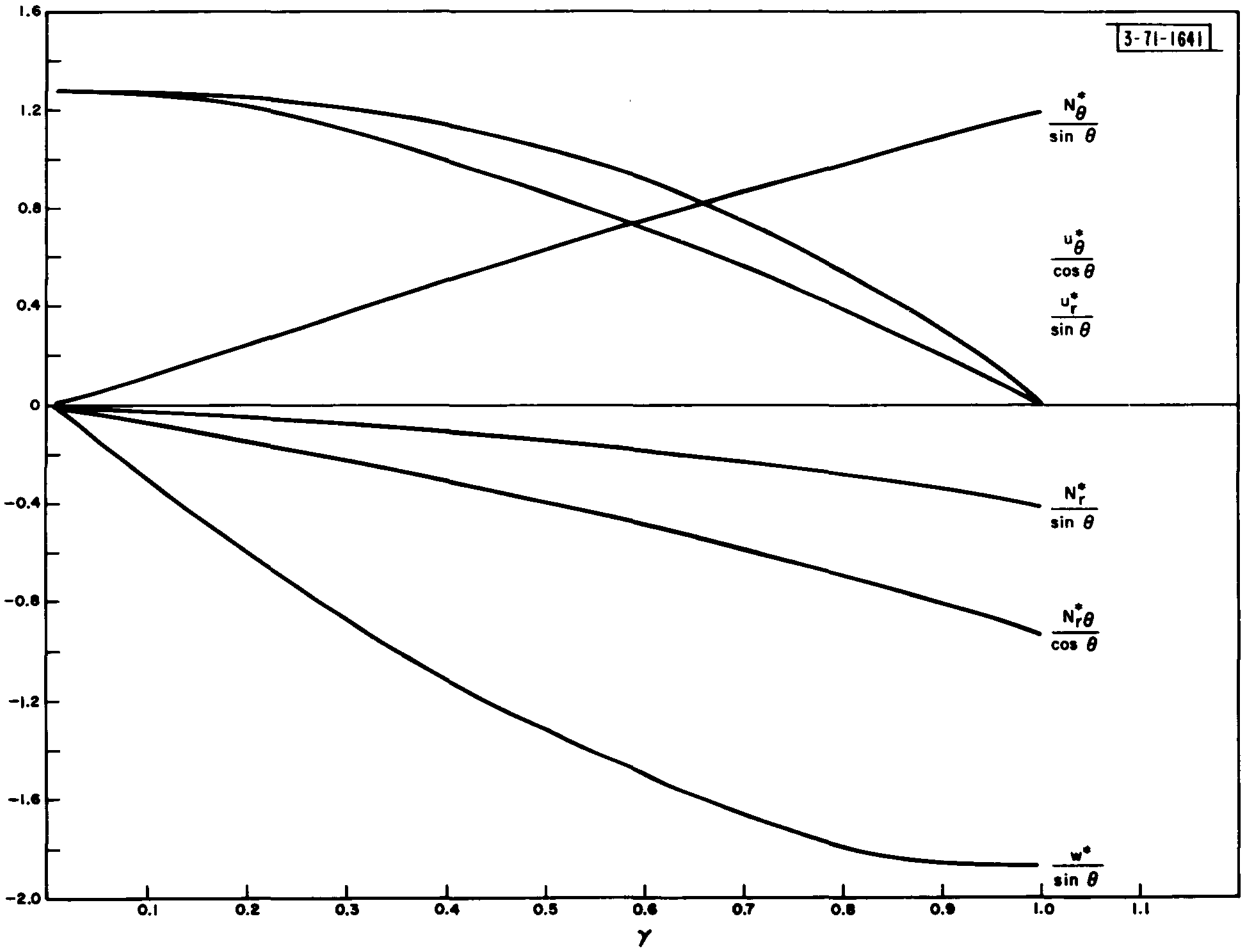
CASE 6.6.2.4 ASYMMETRIC GRAVITY

Free $\gamma_1 = 0.625$

Restrained $\gamma_2 = 0.3854$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.6250	−0.000	0.625	−0.000
0.6010	−0.0013	0.602	0.0584
0.5770	−0.0058	0.581	0.121
0.5531	−0.0141	0.463	0.190
0.5291	−0.0271	0.550	0.265
0.5052	−0.0461	0.541	0.349
0.4812	−0.0727	0.540	0.443
0.4572	−0.109	0.547	0.551
0.4333	−0.158	0.566	0.676
0.4093	−0.224	0.601	0.823
0.3854	−0.313	0.658	1.0002

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.6250	−0.358	0.470	−2.197
0.6010	−0.315	0.435	−2.0733
0.5770	−0.274	0.398	−1.946
0.5531	−0.235	0.359	−1.816
0.5291	−0.197	0.317	−1.683
0.5052	−0.161	0.273	−1.547
0.4812	−0.126	0.226	−1.408
0.4572	−0.0932	0.175	−1.265
0.4333	−0.0612	0.121	−1.117
0.4093	−0.0303	0.0635	−0.964
0.3854	0.000	−0.000	−0.806



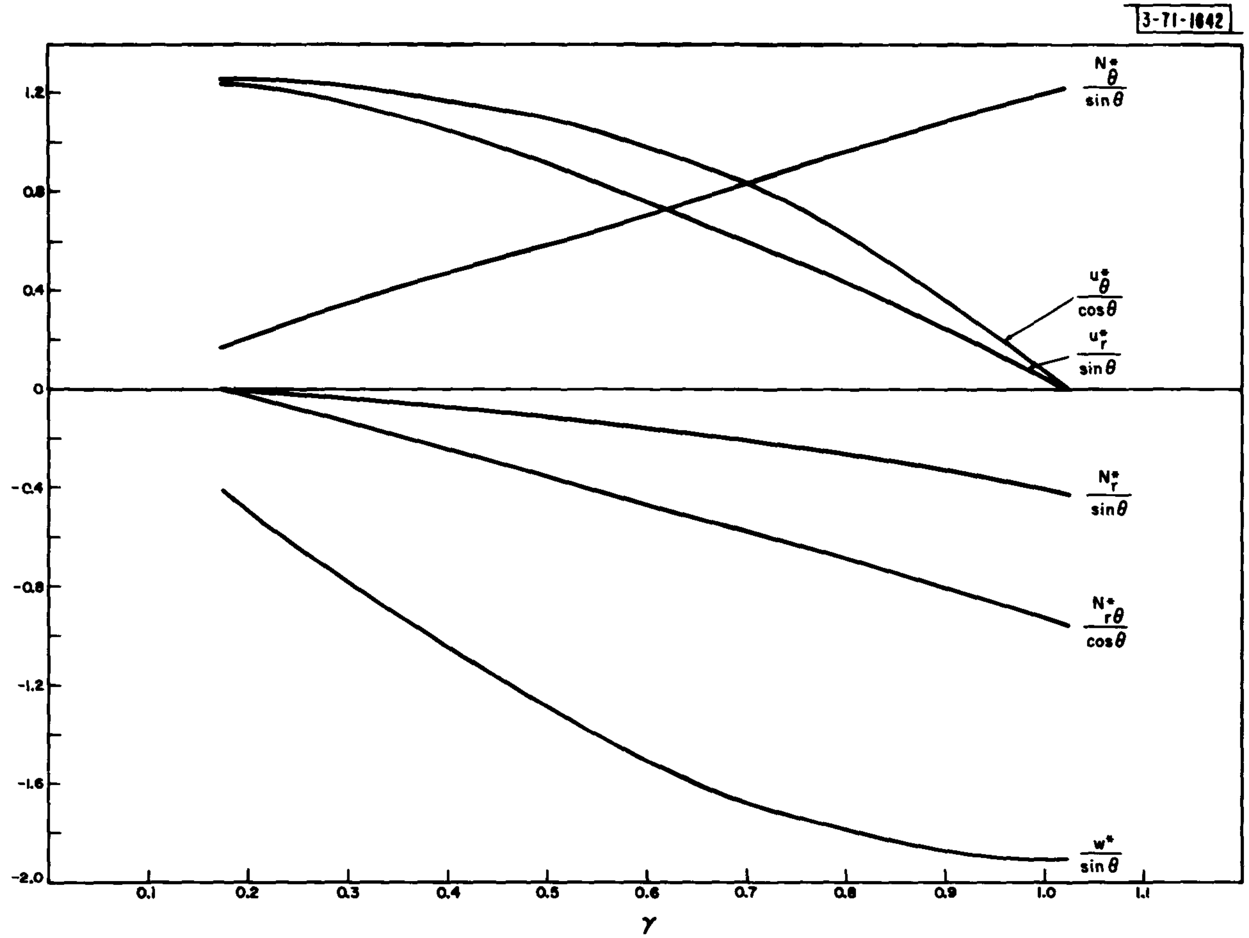
CASE 6.6.2.5 ASYMMETRIC GRAVITY

Free $\gamma_1 = 0.0104$

Restrained $\gamma_2 = 0.9982$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0104	−0.0082	0.0170	0.0065
0.1092	−0.0270	0.135	−0.0816
0.2079	−0.0532	0.2590	−0.157
0.3067	−0.0812	0.381	−0.235
0.4055	−0.112	0.501	−0.317
0.5043	−0.146	0.621	−0.403
0.6031	−0.186	0.739	−0.495
0.7019	−0.230	0.856	−0.593
0.8006	−0.281	0.972	−0.697
0.8994	−0.339	1.0873	−0.809
0.9982	−0.405	1.201	−0.929

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0104	1.280	1.280	−0.0266
0.1092	1.259	1.271	−0.318
0.2079	1.204	1.244	−0.598
0.3067	1.119	1.198	−0.862
0.4055	1.0076	1.131	−1.104
0.5043	0.874	1.039	−1.320
0.6031	0.723	0.917	−1.505
0.7019	0.558	0.759	−1.656
0.8006	0.382	0.558	−1.769
0.8994	0.196	0.308	−1.842
0.9982	0.000	0.000	−1.869



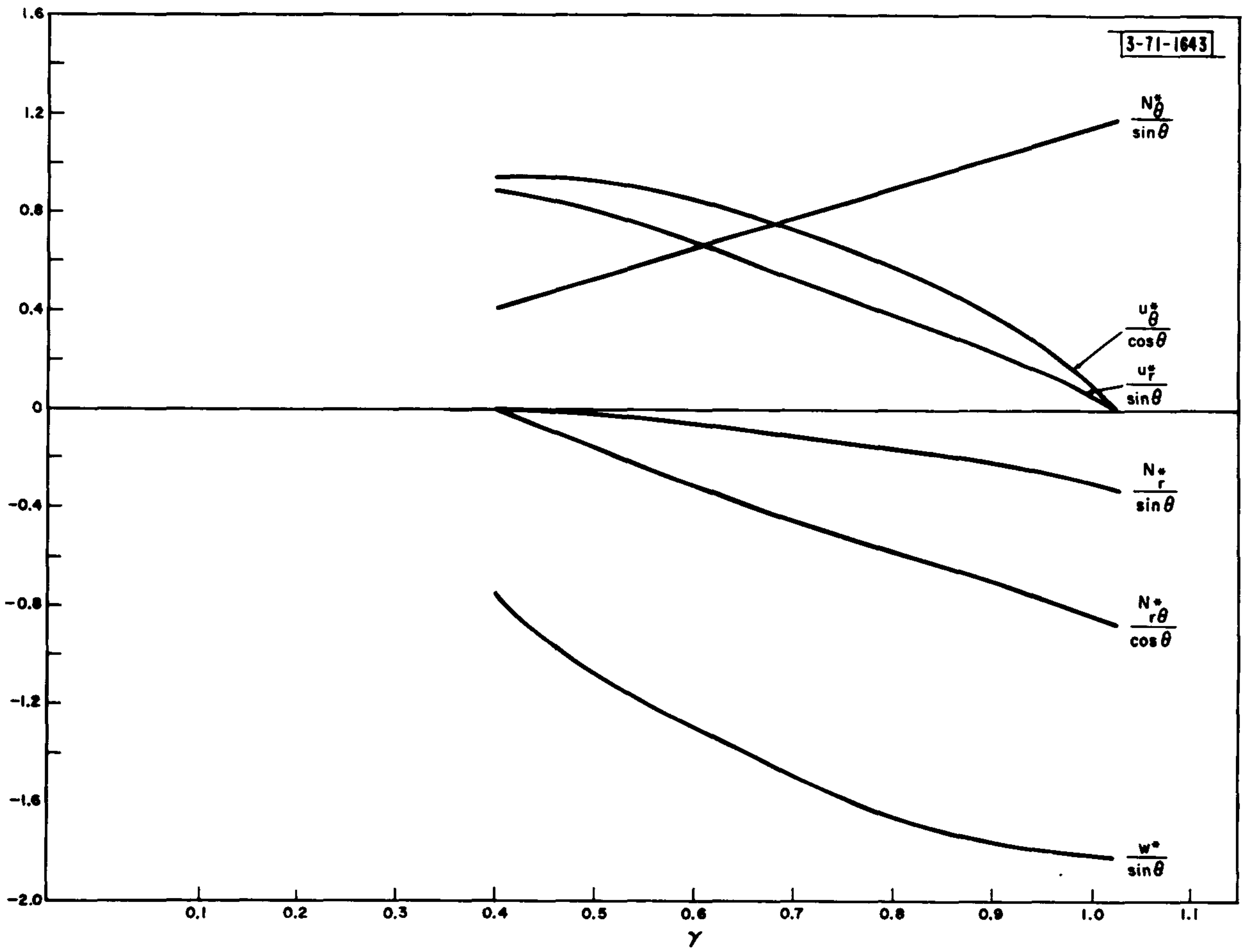
CASE 6.6.2.6 ASYMMETRIC GRAVITY

Free $\gamma_1 = 0.1770$

Restrained $\gamma_2 = 1.0243$

γ	$\frac{N^*_r}{\sin\theta}$	$\frac{N^*_\theta}{\sin\theta}$	$\frac{N^*_{r\theta}}{\cos\theta}$
0.1770	−0.000	0.177	0.000
0.2618	−0.0203	0.280	−0.125
0.3465	−0.0515	0.392	−0.216
0.4312	−0.0844	0.502	−0.299
0.5159	−0.118	0.609	−0.381
0.6006	−0.155	0.715	−0.465
0.6854	−0.196	0.818	−0.552
0.7701	−0.240	0.920	−0.643
0.8548	−0.289	1.021	−0.739
0.9395	−0.342	1.121	−0.840
1.0243	−0.402	1.220	−0.946

γ	$\frac{u^*_r}{\sin\theta}$	$\frac{u^*_\theta}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.1770	1.231	1.251	−0.398
0.2618	1.180	1.238	−0.674
0.3465	1.1058	1.204	−0.918
0.4312	1.0103	1.150	−1.137
0.5159	0.898	1.076	−1.333
0.6006	0.772	0.978	−1.503
0.6854	0.634	0.853	−1.647
0.7701	0.488	0.698	−1.763
0.8548	0.333	0.507	−1.849
0.9395	0.170	0.276	−1.902
1.0243	−0.000	0.000	−1.920



CASE 6.6.2.7 ASYMMETRIC GRAVITY

Free $\gamma_1 = 0.4010$

Restrained $\gamma_2 = 1.0243$

γ	$\frac{N^*_r}{\sin\theta}$	$\frac{N^*_\theta}{\sin\theta}$	$\frac{N^*_{r\theta}}{\cos\theta}$
0.4010	-0.000	0.401	0.000
0.4633	-0.0087	0.470	-0.118
0.5256	-0.0284	0.547	-0.217
0.5880	-0.0539	0.628	-0.305
0.6503	-0.0831	0.708	-0.388
0.7126	-0.115	0.789	-0.468
0.7749	-0.150	0.868	-0.547
0.8373	-0.187	0.947	-0.626
0.8996	-0.227	1.0254	-0.707
0.9619	-0.270	1.102	-0.789
1.0243	-0.316	1.178	-0.874

γ	$\frac{u^*_r}{\sin\theta}$	$\frac{u^*_\theta}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.4010	0.880	0.937	-0.756
0.4633	0.825	0.926	-0.942
0.5256	0.761	0.897	-1.109
0.5880	0.688	0.853	-1.259
0.6503	0.607	0.791	-1.392
0.7126	0.520	0.712	-1.509
0.7749	0.426	0.614	-1.608
0.8373	0.327	0.496	-1.690
0.8996	0.223	0.355	-1.754
0.9619	0.114	0.190	-1.799
1.0243	0.000	0.000	-1.823

6.6.3 THE SHELL IS CLOSED AT THE APEX

The condition of finite stress resultants at the apex requires that

$$C_3 = \frac{-1}{3} \qquad 6.6.3.1$$

and

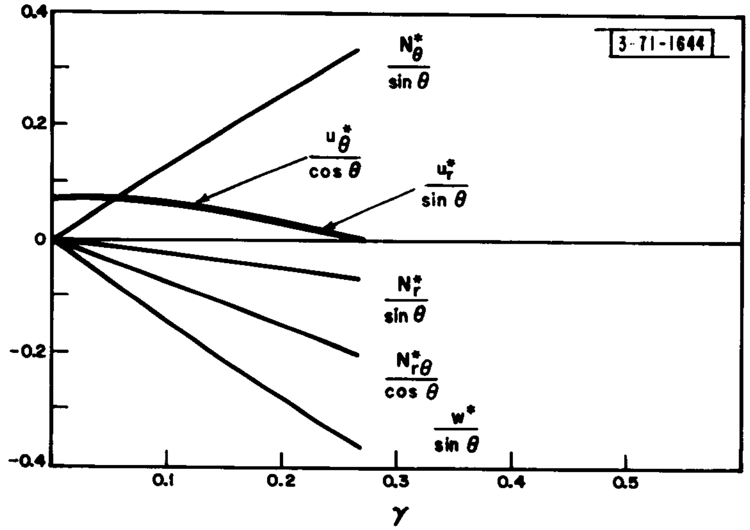
$$C_4 = \frac{-1}{15} \qquad 6.6.3.2$$

C_5 and C_6 can again be obtained from equations (6.6.2.3) and (6.6.2.4).

The six configurations which have been analyzed in this section are summarized in Table 6.6.3.1, and the discussion of section 6.6.1 is applicable. No additional comments appear to be needed.

Table 6.6.3.1

Case	Restrained at γ_2
6.6.3.1	.2673
6.6.3.2	.3854
6.6.3.3	.5381
6.6.3.4	.6250
6.6.3.5	.8038
6.6.3.6	1.0243



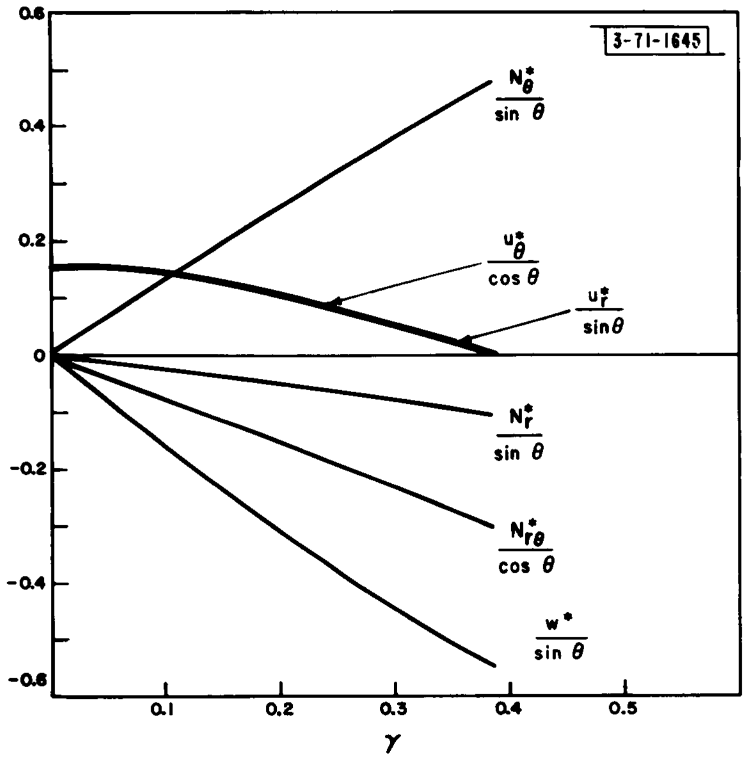
CASE 6.6.3.1 ASYMMETRIC GRAVITY

Closed at Apex $\gamma_1 = 0.0$

Restrained $\gamma_2 = 0.2673$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0	0.0	0.0	−0.0
0.0361	−0.0089	0.0450	−0.0271
0.0618	−0.0154	0.0772	−0.0464
0.0874	−0.0219	0.109	−0.0657
0.1131	−0.0285	0.141	−0.0852
0.1388	−0.0351	0.173	−0.104
0.1645	−0.0418	0.205	−0.124
0.1902	−0.0487	0.237	−0.144
0.2159	−0.0556	0.269	−0.164
0.2416	−0.0627	0.300	−0.184
0.2673	−0.0700	0.332	−0.204

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0	–	–	−0.0
0.0361	0.0713	0.0714	−0.0515
0.0618	0.0687	0.0689	−0.0879
0.0874	0.0647	0.0652	−0.124
0.1131	0.0594	0.0600	−0.160
0.1388	0.0527	0.0536	−0.195
0.1645	0.0446	0.0457	−0.231
0.1902	0.0353	0.0365	−0.265
0.2159	0.0248	0.0258	−0.299
0.2416	0.0130	0.0136	−0.333
0.2673	0.000	0.000	−0.366



CASE 6.6.3.2 ASYMMETRIC GRAVITY

Closed at Apex $\gamma_1 = 0.0$

Restrained $\gamma_2 = 0.3854$

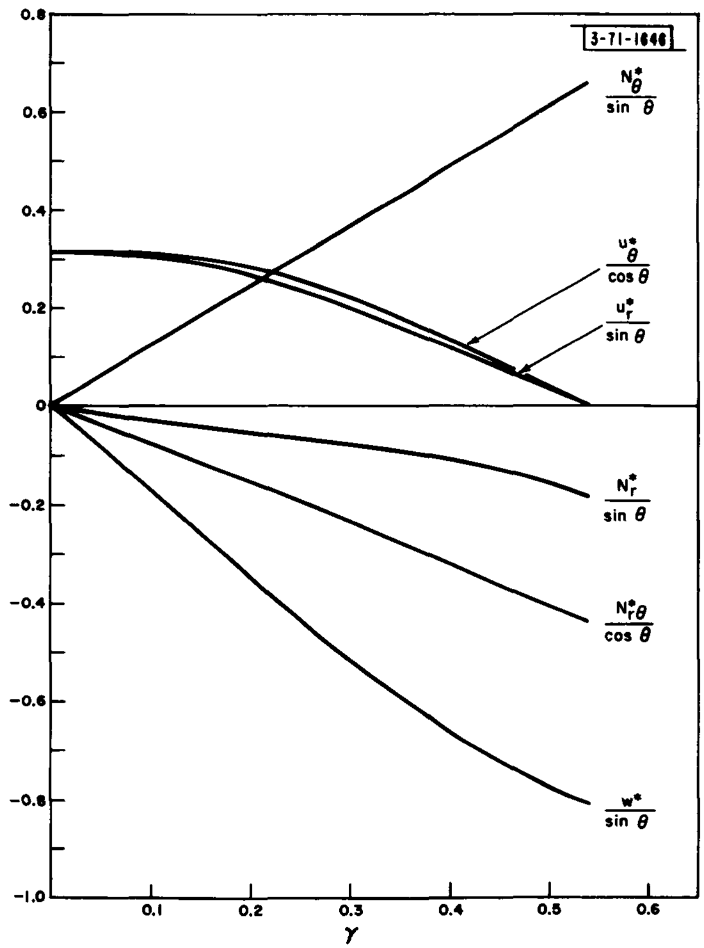
γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0	0.0	0.0	0.0
0.0479	−0.0119	0.0598	−0.0359
0.0854	−0.0214	0.106	−0.0642
0.1229	−0.0310	0.153	−0.0925
0.1604	−0.0407	0.200	−0.121
0.1979	−0.0507	0.246	−0.150
0.2354	−0.0610	0.293	−0.179
0.2729	−0.0715	0.339	−0.208
0.3104	−0.0825	0.385	−0.238
0.3479	−0.0939	0.431	−0.269
0.3854	−0.105	0.477	−0.300

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0	–	–	−0.0
0.0479	0.151	0.152	−0.0735
0.0854	0.146	0.147	−0.130
0.1229	0.137	0.139	−0.187
0.1604	0.126	0.129	−0.242
0.1979	0.111	0.115	−0.297
0.2354	0.0946	0.0992	−0.350
0.2729	0.0747	0.0795	−0.402
0.3104	0.0522	0.0565	−0.452
0.3479	0.0272	0.0301	−0.500
0.3854	0.000	0.000	−0.545

CASE 6.6.3.3 ASYMMETRIC GRAVITY

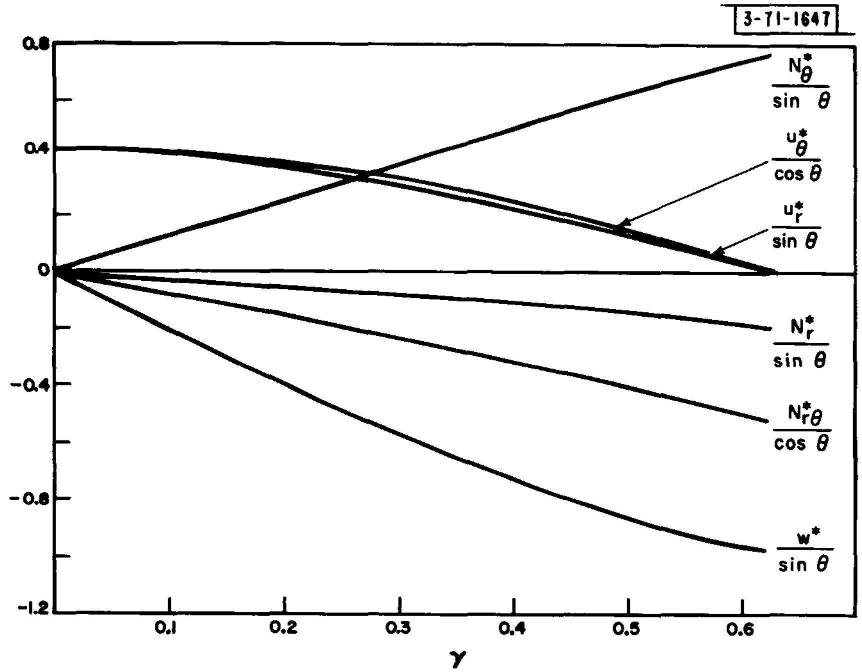
Closed at Apex $\gamma_1 = 0.0$

Restrained $\gamma_2 = 0.5381$



γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0	0.0	0.0	0.0
0.0631	−0.0158	0.0789	−0.0474
0.1159	−0.0292	0.144	−0.6873
0.1687	−0.0429	0.210	−0.127
0.2215	−0.0571	0.276	−0.168
0.2743	−0.0719	0.341	−0.209
0.3270	−0.0875	0.406	−0.252
0.3798	−0.103	0.470	−0.296
0.4326	−0.121	0.534	−0.340
0.4854	−0.140	0.598	−0.386
0.5381	−0.159	0.662	−0.434

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0	–	–	−0.0
0.0631	0.307	0.308	−0.110
0.1159	0.296	0.299	−0.201
0.1687	0.278	0.285	−0.290
0.2215	0.254	0.265	−0.376
0.2743	0.224	0.238	−0.460
0.3270	0.189	0.206	−0.539
0.3798	0.148	0.166	−0.614
0.4326	0.103	0.119	−0.684
0.4854	0.0536	0.0641	−0.748
0.5381	−0.000	−0.000	−0.806



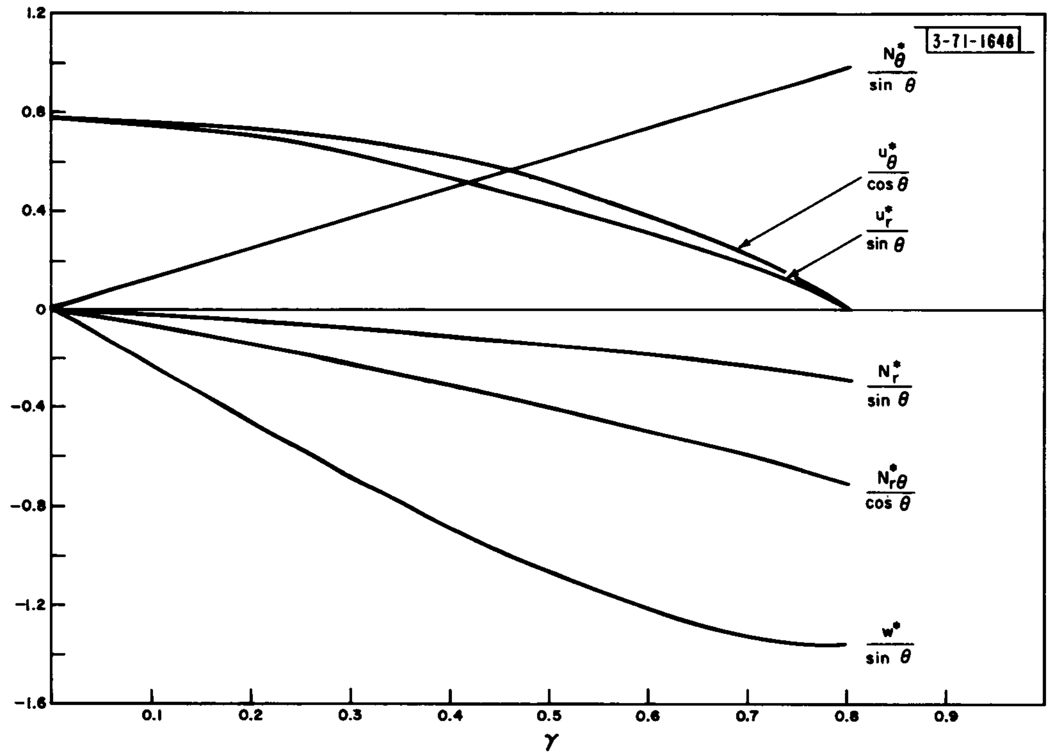
CASE 6.6.3.4 ASYMMETRIC GRAVITY

Closed at Apex $\gamma_1 = 0.0$

Restrained $\gamma_2 = 0.625$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0	0.0	0.0	−0.0
0.0718	−0.0180	0.0898	−0.0539
0.1333	−0.0337	0.166	−0.100
0.1947	−0.049	0.252	−0.147
0.2562	−0.0668	0.318	−0.195
0.3177	−0.0847	0.394	−0.244
0.3791	−0.103	0.469	−0.295
0.4406	−0.124	0.544	−0.347
0.5020	−0.146	0.618	−0.401
0.5635	−0.169	0.692	−0.458
0.6249	−0.195	0.765	−0.516

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0	-0.000	-0.000	-0.000
0.0718	0.426	0.428	−0.136
0.1333	0.410	0.416	−0.251
0.1947	0.384	0.396	−0.363
0.2562	0.350	0.369	−0.471
0.3177	0.308	0.334	−0.573
0.379	0.259	0.289	−0.669
0.4406	0.202	0.235	−0.758
0.5020	0.140	0.169	−0.839
0.5635	0.0727	0.0915	−0.910
0.6249	−0.0	−0.0	−0.0



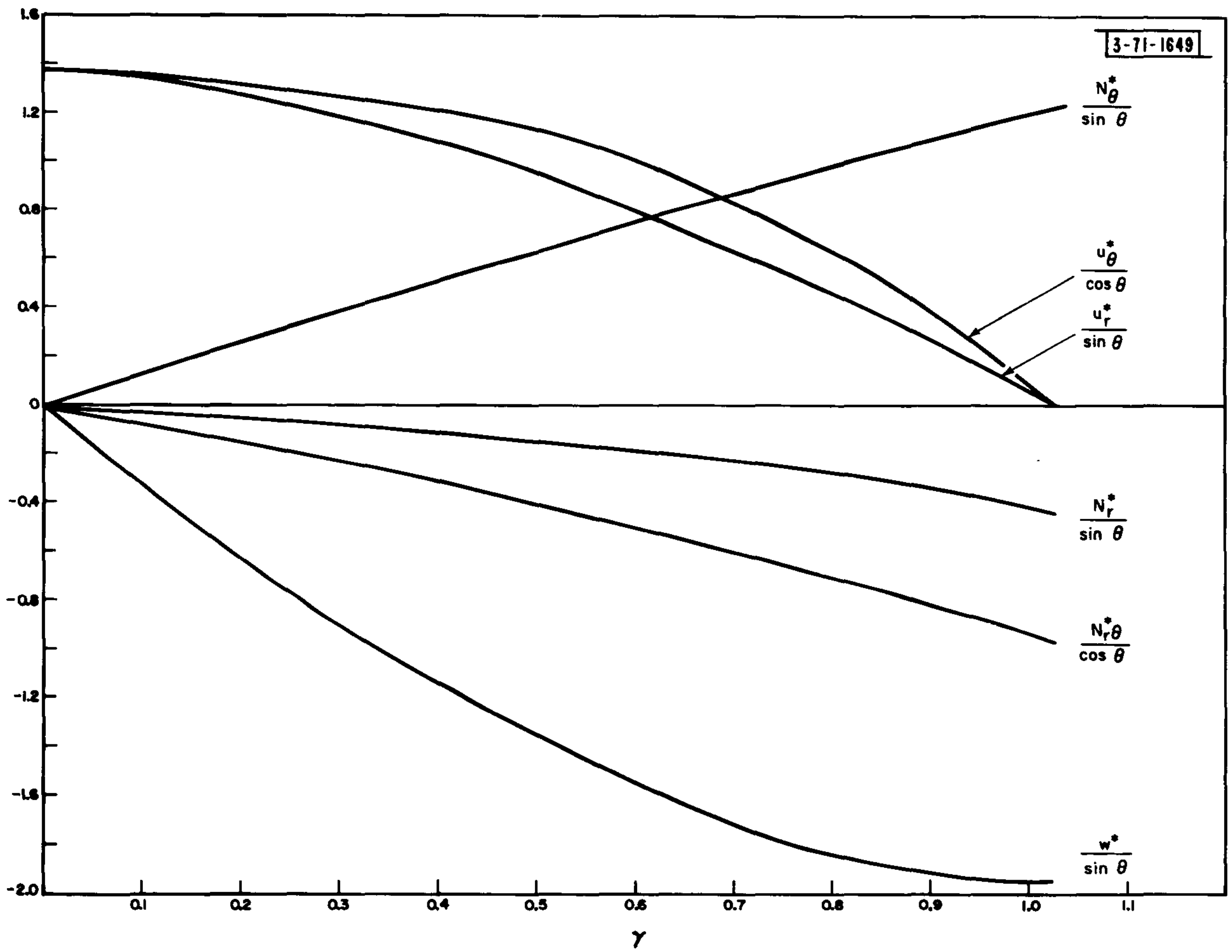
CASE 6.6.3.5 ASYMMETRIC GRAVITY

Closed at Apex $\gamma_1 = 0.0$

Restrained $\gamma_2 = 0.8038$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0	0.0	0.0	-0.0
0.0897	-0.0225	0.112	-0.0674
0.1690	-0.0430	0.210	-0.127
0.2484	-0.0646	0.309	-0.189
0.3277	-0.0877	0.407	-0.253
0.4071	-0.112	0.503	-0.319
0.4864	-0.140	0.600	-0.387
0.5657	-0.170	0.695	-0.460
0.6451	-0.204	0.789	-0.536
0.7244	-0.242	0.883	-0.616
0.8038	-0.283	0.976	-0.701

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0	–	–	-0.0
0.0897	0.751	0.756	-0.207
0.1690	0.721	0.738	-0.386
0.2484	0.674	0.707	-0.558
0.3277	0.611	0.663	-0.720
0.4071	0.534	0.604	-0.870
0.4864	0.445	0.528	-1.0057
0.5657	0.346	0.432	-1.124
0.6451	0.238	0.315	-1.224
0.7244	0.122	0.172	-1.304
0.8038	0.000	0.000	-1.361



CASE 6.6.3.6 ASYMMETRIC GRAVITY

Closed at Apex $\gamma_1 = 0.0$

Restrained $\gamma_2 = 1.0243$

γ	$\frac{N_r^*}{\sin\theta}$	$\frac{N_\theta^*}{\sin\theta}$	$\frac{N_{r\theta}^*}{\cos\theta}$
0.0	0.0	0.0	-0.0
0.1118	-0.0281	0.139	-0.0841
0.2131	-0.0549	0.265	-0.161
0.3145	-0.0837	0.390	-0.242
0.4159	-0.115	0.514	-0.326
0.5173	-0.151	0.637	-0.415
0.6187	-0.193	0.758	-0.510
0.7201	-0.239	0.878	-0.611
0.8215	-0.293	0.996	-0.720
0.9229	-0.354	1.114	-0.837
1.0243	-0.424	1.231	-0.962

γ	$\frac{u_r^*}{\sin\theta}$	$\frac{u_\theta^*}{\cos\theta}$	$\frac{w^*}{\sin\theta}$
0.0	–	–	-0.0
0.1118	1.342	1.355	-0.337
0.2131	1.283	1.327	-0.633
0.3145	1.191	1.279	-0.912
0.4159	1.0714	1.209	-1.167
0.5173	0.928	1.111	-1.392
0.6187	0.767	0.981	-1.585
0.7201	0.592	0.813	*-1.740
0.8215	0.504	0.599	-1.854
0.9229	0.208	0.331	-1.924
1.0243	-0.000	0.000	-1.944

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